On the minimum edge size for 2-colorability and realizability of hypergraphs by axis-parallel rectangles

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Abstract

Given a hypergraph $\mathsf{H} = (\mathcal{V}, \mathcal{E})$ what is the minimum integer $\lambda(\mathsf{H})$ such that the sub-hypergraph with edges of size at least $\lambda(\mathsf{H})$ is 2-colorable? We consider the computational problem of finding the smallest such integer for a given hypergraph, and show that it is NP-hard to approximate it to within $\log m$ where $m = |\mathcal{E}|$. For most geometric hypergraphs, i.e., those defined on a set of n points by intersecting it with some shapes, it is well known that there is a coloring with 2 colors 'red' and 'blue', such that any hyperedge containing $c \log n$ points, for some constant c, is bi-chromatic, i.e., contains points of both colors. We observe that indeed, for several such hypergraph families, this is the best possible – i.e., there are some n points where there will always be a hyperedge with $\Omega(\log n)$ points that is mono-chromatic. These results follow from results on the indecomposability of coverings. We also show that a frequently used hypergraph, used in the literature on indecomposable coverings *cannot* be realized by axis-parallel rectangles in the plane. This problem was mentioned in a paper of Pach et al. on indecomposable coverings.

1 Introduction

Given a hypergraph $\mathsf{H} = (\mathcal{V}, \mathcal{E})$, suppose every element of $\mathcal{V}$ is assigned one of two colors 'red' and 'blue'. An edge $e \in \mathcal{E}$ is properly colored if not all elements of e receive the same color. The property of being 2-colorable is a well studied problem in combinatorics. It is also known as "Property B", a term coined by E.W. Miller [12] in honor of Felix Bernstein [1]. For many geometric hypergraphs, i.e., hypergraphs defined by a set of n points by intersecting it with some geometric shapes, it is known that the sub-hypergraph of all edges of size at least $t \geq c \log n$ for some constant c has property B. A natural question is to ask what is the smallest possible value of t with this property? Denote this minimum by $\lambda(\mathsf{H})$. In this paper we investigate this question, with a focus on some geometric hypergraphs.

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[1]Sometimes, Property B is only defined for hypergraphs where each edge has the same size, but we use it for general hypergraphs.

Property B. A simple probabilistic argument of Erdős[4] shows that if each edge size is at least p, and the hypergraph has at most 2^{p-1} edges, it is 2-colorable. Erdős and Hajnal [5] asked: What is the smallest number of edges in a hypergraph each of which have size p, without Property B. They showed that $m(p) < 4^p$ [2]. In geometric hypergraphs on n points in $\mathbb{R}^d$, bounded VC-dimension arguments [10] often mean that the total number of edges is at most $O(n^d)$. Thus, if $p \approx c \log n$, for some constant c, since 2^{p-1} exceeds the number of edges, the sub-hypergraph with edge size at least p has property B.

Cover decomposability. The problem we study is related to the concept of cover decomposability, a topic that has been the subject of much recent research, see the survey [15]. Consider a family of sets $\mathcal{S}$ in the plane (or space). Now, given a finite set of points P, one can define a *primal* hypergraph $\mathsf{H} = (P, \mathcal{R})$ by intersecting the sets in $\mathcal{S}$ with P. Likewise, one can define a dual hypergraph H^* by restricting to a finite sub-family of $\mathcal{S}$, and an edge for each point of the plane (or space) defined by the sets in the sub-family that contain it. In cover decomposability one is interested in the problem : Is $\lambda(\mathsf{H}^*) = O(1)$? A result of Pach [13] implies that if one is looking at the family of translates of an open convex polygon, the problems for the primal and dual hypergraphs are equivalent, i.e., $\lambda(\mathsf{H}) = O(1)$ iff $\lambda(\mathsf{H}^*) = O(1)$. There are many positive and negative results known about cover decomposability, for example, the family of all translates of a given open convex polygon is cover decomposable [18], disks and other convex shapes with a smooth boundary are not cover decomposable [14], but if the convex set is unbounded then it is cover decomposable; translates of concave polygons (most of them) are known to be not cover decomposable [17]. The family of all homothets of a triangle is cover decomposable [6, 7] but homothets of any convex polygon with at least four sides are not cover decomposable [8]. The family of axis-parallel rectangles is also not cover decomposable [16]. The problem of when $\lambda(\mathsf{H})$ (i.e., for the primal hypergraphs) is $O(1)$ has also received recent attention - for example, it has been recently shown to be true for squares [1], while it is known to be false for axis-parallel rectangles [3]. As mentioned

[2]Better bounds are now known.

before, results for the dual setting also imply results for the primal setting for families which are translates of open convex polygons. Techniques for proving indecomposability of coverings are very useful for lower bounding $\lambda(\mathsf{H})$. Typically, this is done by proving that a certain hypergraph which is not 2-colorable can be realized by the dual (or primal) geometric hypergraph.

Our contributions. Our contributions can be summarized as follows:

(I) We show that for a given hypergraph $\mathsf{H} = (\mathcal{V}, \mathcal{E})$ it is NP-hard to approximately compute $\lambda(\mathsf{H})$ up-to a factor of $\log m$, where $m = |\mathcal{E}|$.

(II) We show that a certain frequently used hypergraph, which has been used before in proving indecomposability of coverings, cannot be realized by axis-parallel rectangles. This problem was mentioned in a paper of Pach et al. [16].

Paper organization. In Section 2 we provide definitions, set up notation, and provide some simple results on $\lambda(\mathsf{H})$. We prove the computational hardness of approximating $\lambda(\mathsf{H})$ in Section 3. In Section 4 we observe that known constructions from the indecomposability of coverings literature imply lower bounds on $\lambda(\mathsf{H})$ for some primal and dual geometric hypergraphs. In Section 5 we prove our result on the non-realizability of a hypergraph from [16] as the primal hypergraph of axis-parallel rectangles.

2 Preliminaries and Basic Results

A hypergraph $\mathsf{H} = (\mathcal{V}, \mathcal{E})$ is a finite set $\mathcal{V}$ along with a collection $\mathcal{E}$ of subsets of $\mathcal{V}$. Given a hypergraph $\mathsf{H} = (\mathcal{V}, \mathcal{E})$, and an integer t with $0 \le t \le |\mathcal{V}|$, define the t-level hypergraph H_t to be the hypergraph $(\mathcal{V}, \mathcal{E}_t)$ where $\mathcal{E}_t = \{e \in \mathcal{E} \mid |e| \ge t\}$. We define the *property B threshold*, $\lambda(\mathsf{H})$, to be the minimum integer t with $2 \le t \le |\mathcal{V}|$ such that $(\mathcal{V}, \mathcal{E}_t)$ has property B. For a hypergraph $\mathsf{H} = (\mathcal{V}, \mathcal{E})$ where $|e| = k \ge 2$ for every $e \in \mathcal{E}$ we have $\lambda(\mathsf{H}) = 2$ iff $(\mathcal{V}, \mathcal{E})$ has property B. [3]

A family of hypergraphs. We often work with families of hypergraphs, which are collections of "related" hypergraphs, such that for every integer n we have (possibly several) hypergraphs $(\mathcal{V}, \mathcal{E})$ in the family with $|\mathcal{V}| = n$. In such cases, we will be interested in deriving lower bounds on $\lambda(\mathsf{H})$ as a function of n – such a bound will be worst case, i.e., for almost all values of n, there will be some hypergraph H with n vertices in the family with $\lambda(\mathsf{H})$ lower bounded by the function. Some important families are defined geometrically and are of special interest to us. Let $\mathcal{S}$ be a family of sets in $\mathbb{R}^d$ (for example, all axis-parallel boxes, or balls, or convex polytopes etc.) Two families can be defined as follows:

(A) **Primal geometric family.** A hypergraph in the family is defined by a set P of n points as $\mathsf{H} = (P, \mathcal{E})$ where the edges are defined by intersecting P with members of $\mathcal{S}$.

(B) **Dual geometric family.** A hypergraph H^* in the family is defined by a finite sub-family $\mathcal{S}^*$ of $\mathcal{S}$; $\mathsf{H}^* = (\mathcal{S}^*, \mathcal{E}^*)$ where for each point p of space, an edge $e^* \in \mathcal{E}^*$ is defined by looking at all the sets in $\mathcal{S}^*$ that contain p.

We now introduce some hypergraphs which are not 2-colorable. They have been used previously to prove indecomposability of coverings in [16, 17, 14]. Given a rooted tree $T = (V, E)$, define a hypergraph $\mathsf{H}(T) = (\mathcal{V}(T), \mathcal{E}(T))$, where $\mathcal{V}(T) = V$ and the set of edges $\mathcal{E}(T)$ is the union $\mathcal{E}_L(T) \cup \mathcal{E}_S(T)$. An edge in $\mathcal{E}_L(T)$ is the set of vertices on a root-to-leaf path, and an edge of $\mathcal{E}_S(T)$ is the set of vertices which are the children of some internal node. Observe that the number of edges of edges is precisely $|\mathcal{V}(T)| = |V|$. For a given integer $k \ge 2$, let T_k denote the rooted tree where every internal node has degree k and the height is $k-1$. It is clear that the hypergraph $\mathsf{H}(T_k)$ is k-regular and it has $\frac{k^k-1}{k-1}$ vertices and edges.

Another commonly used hypergraph is $\mathsf{H}(k, \ell) = (\mathcal{V}(k, \ell), \mathcal{E}(k, \ell))$ for integers $k, \ell \ge 1$ [17, 14]. We do not need its exact definition here, but mention some basic properties relevant to us. The edge set $\mathcal{E}(k, \ell)$ is a disjoint union of 'red' edges and 'blue' edges where each red edge has k elements and each blue edge has ℓ elements. The hypergraph has the property that in any 2 coloring of $\mathcal{V}(k, \ell)$ with colors 'red' and 'blue' either, (1) all vertices in some red edge are colored red, or (2) all vertices in some blue edge are colored blue. The hypergraph $\mathsf{H}(k, \ell)$ has $|\mathcal{V}(k, \ell)| = \binom{k+\ell}{k} - 1$ while $|\mathcal{E}(k, \ell)| = \binom{k+\ell}{k}$. If $\ell = k$, the number of edges is $\binom{2k}{k} \le 4^k$. We say that we can realize a family of hypergraphs $\mathsf{H}_n = (\mathcal{V}_n, \mathcal{E}_n)$ as the primal (resp. dual) hypergraph of a family of sets $\mathcal{S}$, if for each integer $t \ge 1$ and each hypergraph $\mathsf{H} = (\mathcal{V}, \mathcal{E})$ in the family with $|\mathcal{V}| = t$ there is a set of points P in the plane such that the primal (resp. dual) hypergraph induced by P and $\mathcal{S}$ is isomorphic to H.

2.1 Elementary properties of $\lambda(\mathsf{H})$

The following are elementary and we omit their proof.

Lemma 1 *For a hypergraph $\mathsf{H} = (\mathcal{V}, \mathcal{E})$ we have that $\lambda(\mathsf{H}) \le 2\log(2|\mathcal{E}| + 1)$.*

Lemma 2 *For a hypergraph $\mathsf{H} = (\mathcal{V}, \mathcal{E})$, $\lambda(\mathsf{H}) \le \mathsf{disc}(\mathsf{H}) + 1$, where $\mathsf{disc}(\mathsf{H})$ is the combinatorial discrepancy.*

[3] Notice that we do not require that there is an edge $e \in \mathcal{E}$ with $|e| = \lambda(\mathsf{H})$. For example, the hypergraph with a single edge of large cardinality obviously has property B, and we have $\lambda(\mathsf{H}) = 2$, but there is no edge of size 2.

3 Computational Hardness Result

It is NP-hard to decide if a given k-uniform hypergraph is 2-colorable for $k \geq 3$ [9]. As such given a hypergraph $\mathsf{H} = (\mathcal{V}, \mathcal{E})$ deciding if $\lambda(\mathsf{H}) = 2$ is NP-complete and thus it is NP-hard to compute $\lambda(\mathsf{H})$ exactly. We show here that it is NP-hard to approximate it within a factor $\log m$ where $m = |\mathcal{E}|$.

Theorem 3 *There is a constant $c > 0$ such that the following problem is NP-hard: Given a hypergraph $\mathsf{J} = (\mathcal{U}, \mathcal{F})$ with $|\mathcal{F}| = m$ output a number α with $\lambda(\mathsf{J}) \leq \alpha \leq c \log m \cdot \lambda(\mathsf{J})$.*

Proof. Consider an instance of determining whether a given 4-uniform hypergraph $\mathsf{H} = (\mathcal{V}, \mathcal{E})$ is 2-colorable. Let $\mathcal{E} = \{e_1, e_2, \ldots, e_s\}$ and $|e_i| = 4$ for $1 \leq i \leq s$. Let $k = 2\lceil \log s \rceil$ (an even number). The hypergraph $\mathsf{H}' = \mathsf{H}(k,k) = (\mathcal{V}', \mathcal{E}')$, where $\mathcal{V}'$ is disjoint from $\mathcal{V}$, has at most $4^k \leq s^5$ edges for s large enough. Let $\mathsf{H}'' = (\mathcal{V}'', \mathcal{E}'')$ be defined on distinct elements but isomorphic to H'. Now consider the following three hypergraphs: (1) $\mathsf{H}_1 = (\mathcal{V} \cup \mathcal{V}', \mathcal{E}_1)$ where $\mathcal{E}_1 = \{e_1 \cup e_2 \mid e_1 \in \mathcal{E}, e_2 \in \mathcal{E}'\}$,

(2) $\mathsf{H}_2 = (\mathcal{V} \cup \mathcal{V}'', \mathcal{E}_2)$ where $\mathcal{E}_2 = \{e_1 \cup e_2 \mid e_1 \in \mathcal{E}, e_2 \in \mathcal{E}''\}$, and, (3) $\mathsf{H}_3 = (\mathcal{V}' \cup \mathcal{V}'', \overline{\mathcal{E}})$, where $\overline{\mathcal{E}}$ is defined as follows: For each $e' \in \mathcal{E}'$, choose arbitrarily $k/2$ elements of e' and call this set $\overline{e'}$. Similarly, define $\overline{e''}$ for each $e'' \in \mathcal{E}''$. Now, $\overline{\mathcal{E}} = \{\overline{e'} \cup \overline{e''} \mid e' \in \mathcal{E}', e'' \in \mathcal{E}''\}$.

Lemma 4 *The hypergraph $\mathsf{J} = (\mathcal{V} \cup \mathcal{V}' \cup \mathcal{V}'', \mathcal{E}_1 \cup \mathcal{E}_2 \cup \overline{\mathcal{E}})$ is 2-colorable iff $(\mathcal{V}, \mathcal{E})$ is 2-colorable.*

Proof. If $(\mathcal{V}, \mathcal{E})$ is 2-colorable, then choose the same coloring for elements of $\mathcal{V}$, color all elements of $\mathcal{V}'$ blue and all elements of $\mathcal{V}''$ red. It is easy to see that all the edges of $\mathcal{E}_1 \cup \mathcal{E}_2 \cup \overline{\mathcal{E}}$ are properly colored.

On the other hand, suppose that J can be 2-colored. Since $\mathcal{V}, \mathcal{V}', \mathcal{V}''$ are mutually disjoint one can look upon the coloring as a coloring on the three of those sets separately. We claim that the coloring colors $(\mathcal{V}, \mathcal{E})$ properly. Suppose, this is not true. Then there is an edge $e \in \mathcal{E}$ that is mono-chromatic, say all its elements are blue. Since we know that H' cannot be 2-colored there is some $e' \in \mathcal{E}'$ which is mono-chromatic. The color of e' cannot be blue otherwise the edge $e \cup e' \in \mathcal{E}_1$ would be all blue. So it must be red. Similarly there is a $e'' \in \mathcal{E}''$ that is all red. However then, $\overline{e'} \cup \overline{e''} \in \overline{\mathcal{E}}$ is all red and this contradicts the fact that the coloring is proper for J which includes this edge. It must therefore be true that $\mathcal{E}$ has been properly 2-colored by the induced coloring. □

For the hypergraph J, the total number of edges m is clearly polynomial in s each of cardinality at most $k + 4 = \Theta(\log s)$. Thus the total size of the new hypergraph is polynomial in s. Also, $\log m = O(\log s)$, where m is the number of edges of J. In particular, there is a constant c such that $2c \log m < k$.

Now, by Lemma 4, if $\lambda(\mathsf{H}) = 2$ then $\lambda(\mathsf{J}) = 2$ otherwise $\lambda(\mathsf{J}) \geq k$, since all edge sizes are at least k in J. Suppose we have an approximation algorithm that approximates $\lambda(\mathsf{J})$ to within $c \log m$, i.e., it outputs an α with $\lambda(\mathsf{J}) \leq \alpha \leq c \log m \cdot \lambda(\mathsf{J})$. Then, we can decide if H is 2-colorable as follows: If $\alpha \leq 2c \log m$, then output that H is 2 colorable, otherwise it is not. One can verify easily that the algorithm will correctly decide 2 colorability of H. The reduction is complete. □

In light of the above and Lemma 1, the algorithm which simply outputs $O(\log |\mathcal{E}|)$ as an approximation for $\lambda(\mathsf{H})$, is asymptotically an optimal approximation algorithm, assuming P≠NP.

4 Lower bound for some Geometric Hypergraphs.

Suppose that we can geometrically realize the graph $\mathsf{H}(T_k)$, for all large enough k, for some family of sets $\mathcal{S}$ as its primal (resp. dual) hypergraph. Then, for the family of hypergraphs induced by $\mathcal{S}$, it follows that $\lambda(\mathsf{H}) = \Omega(\frac{\log n}{\log \log n})$ (resp. $\lambda(\mathsf{H}^*) = \Omega(\log n / \log \log n)$), since for all large enough k, there is some hypergraph induced by $\mathcal{S}$ on an $n = \frac{k^k - 1}{k - 1}$ point set P (resp. with a sub-family of size n) such that for any two coloring of P (resp. sets in the sub-family) some edges with size at least $k = \Omega(\frac{\log n}{\log \log n})$ are mono-chromatic. Similarly, if for all k, ℓ sufficiently large, we can realize $\mathsf{H}(k, \ell)$ then it will follow (letting $\ell = k$ and reasoning as before) that $\lambda(\mathsf{H}) = \Omega(\log n)$ (or $\lambda(\mathsf{H}^*) = \Omega(\log n)$). Results from the literature on indecomposability of coverings imply the following. In each case a realization of all $\mathsf{H}(T_k)$ or all $\mathsf{H}(k, \ell)$ has been shown. The following theorem summarizes such known constructions exhibiting lower bounds on $\lambda(\mathsf{H})$ (resp. $\lambda(\mathsf{H}^*)$) as a function of n where H (resp. H^*) is a geometric primal (resp. dual) hypergraph induced by a set of points P with $|P| = n$ (resp. induced by a sub-family of size n):

Theorem 5 *For the family of, (i) Translates of (open or closed) concave polygons with no parallel sides: $\lambda(\mathsf{H}), \lambda(\mathsf{H}^*) = \Omega(\log n)$ [17]. (ii) Open unit disks: $\lambda(\mathsf{H}), \lambda(\mathsf{H}^*) = \Omega(\log n)$ [14]. (iii) Homothets of any convex polygon with at least 4 sides, or a concave one with no parallel sides: $\lambda(\mathsf{H}^*) = \Omega(\log n)$ [8]. (iv) Open strips: $\lambda(\mathsf{H}), \lambda(\mathsf{H}^*) = \Omega(\log n / \log \log n)$ [16].*

The result in part (ii) of the above theorem, extends to a larger family, all (open or closed) disks as well.

We conjecture that for hypergraphs defined by the family of all axis-parallel rectangles we have $\lambda(\mathsf{H}) = \Omega(\log n)$. Interestingly, by Lemma 2 this would also imply a $\Omega(\log n)$ lower bound for the combinatorial discrepancy of axis-parallel rectangles, which is Tusnády's problem in the plane, see [11]. This bound is already known from [2].

5 Non realizability of $\mathsf{H}(T_k)$ by axis-parallel rectangles

Pach et al. [16] mention (see page 6) that it was not known if the graph $\mathsf{H}(T_k)$ could be realized as the primal hypergraph of the family of axis-parallel rectangles. They were interested in the question if $\lambda(\mathsf{H}) = O(1)$ where H is from the family of primal hypergraphs of axis-parallel rectangles. It was shown in [3] that $\lambda(\mathsf{H}) \neq O(1)$. Here, we answer the question left undecided in [16]. We show that $\mathsf{H}(T_k)$ cannot be realized by axis-parallel rectangles in the plane, for sufficiently large k. We conjecture that $\mathsf{H}(k, \ell)$ is also not realizable (for large enough k, ℓ), and a proof similar to one we give below will probably suffice to show this.

In what follows, whenever we say rectangle we mean an axis-parallel rectangle. Let $\mathsf{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph and $\mathcal{V}' \subseteq \mathcal{V}, \mathcal{E}' \subseteq \mathcal{E}$. A sub-hypergraph H' of H defined by $\mathcal{V}', \mathcal{E}'$ is the hypergraph $(\mathcal{V}', \{e \cap \mathcal{V}' | e \in \mathcal{E}'\})$. We show that for all sufficiently large k, $\mathsf{H}(T_k)$ cannot be realized as the hypergraph of points induced by rectangles. The following is a simple observation, that follows by deleting rectangles and points given a realization of H.

Observation 6 *Let $\mathsf{H} = (\mathcal{V}, \mathcal{E})$ be a hypergraph that can be realized by points wrt rectangles, and let H' be a sub-hypergraph of H. Then H' can also be realized by points wrt rectangles.*

Another easy observation is the following.

Observation 7 *$\mathsf{H}(T_k)$ is a sub-hypergraph of $\mathsf{H}(T_m)$ for all $m \geq k$.*

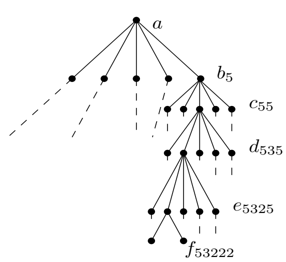

Figure 1: Some vertices and edges of T_0. The height is 5 and each internal node except those with depth 4 have 5 children each. Each node at depth 4 has two leaves as children. The naming of a few vertices is shown.

We omit the easy proof of the following lemma.

Lemma 8 *Let T be an arbitrary rooted tree. Then $\mathsf{H}(T)$ is a sub-hypergraph of $\mathsf{H}(T_k)$ for some k. Moreover, by Observation 7 above, it is a sub-hypergraph for $\mathsf{H}(T_k)$ for all large enough k.*

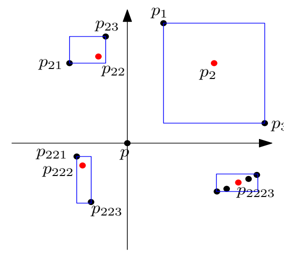

Figure 2: Some of the points shown are forced to be placed as above. The blue rectangles shown are contained in R, R^2, R^{22}, R^{222}.

Now, in order to prove that $\mathsf{H}(T_k)$ cannot be realized by points wrt rectangles, we consider the tree T_0 shown in Figure 1 and we show that there is no way to realize $\mathsf{H}(T_0)$ using rectangles.

For two points p, q in the same quadrant, we say p dominates q if any rectangle containing p and the origin, must contain q. For a set of points in the same quadrant, if none dominates any other, we say they are *on the skyline*.

Theorem 9 *For the tree T_0, $\mathsf{H}(T_0)$ cannot be realized by points wrt rectangles.*

Proof. The root is named a. The vertices at depth 1 are named as $b_1, \ldots, b_5$, those at depth 2 are named c_{ij}, where c_{ij} for fixed i are the children of b_i. Continuing, those at depth 3 are d_{ijk}, at depth 4 are e_{ijkl}. The nodes at depth 6 are f_{ijklm}. Here the indexes i, j, k, l vary from 1 to 5 while m varies in $1, 2$. The proof is by contradiction. Assume that there is a realization of $\mathsf{H}(T_0)$ by points wrt rectangles. Let the point corresponding to the root be p and let p_i be the points corresponding to b_i, p_{ij} for c_{ij} and so on. In the rest of the proof we talk of the points as if they are the vertices of the tree itself with the parent-child or sibling relationships, for brevity. For example, we will say p_{23} is a child of p_2 etc.

Let p be the origin. We may assume that the x and y coordinates of all the points are distinct and the rectangles that define the edges of $\mathsf{H}(T_0)$ contain the relevant points in their interior (as this can be ensured by infinitesimal shifts). The proof will proceed by forcing some points of depth 1 into the 1st quadrant (this is without loss of generality – the crucial point is that they are in the *same* quadrant), then some at depth 2 are forced into the 2nd quadrant etc. Ultimately, we run out of quadrants for the points at depth 5.

We let R denote the rectangle containing the hyperedge $\{p_1, p_2, p_3, p_4, p_5\}$ and in general the children of point, say p_{23}, which define a hyperedge in $\mathcal{E}_S(T_0)$ is realized by rectangle R^{23}. Now, for a hyperedge in $\mathcal{E}_L(T_0)$ defined by say the root-to-leaf path to p_{12445}, let rectangle R_{12445} realize it.

The following lemma follows directly from definitions and the pigeon-hole principle.

Lemma 10 *Let X be a set of points in the plane, $|X| \geq 5$. Suppose that there is a rectangle R_X containing all points in X but not the origin, and for each point $x \in X$, there is a rectangle R_x containing x and the origin, but no other point of X. Then, there are at least 3 points in X that lie in 1 quadrant and are on the skyline.*

Applying Lemma 10 with X as the children of p, i.e., $X = \{p_1, \ldots, p_5\}$, and $R_X = R$ while the rectangles R_x can be taken as any rectangle realizing one of the root-to-leaf hyperedges through one of the p_i we conclude that some three points among $p_1, \ldots, p_5$ lie in one quadrant, which is without loss of generality the first quadrant. Assume these are p_1, p_2, p_3 and that p_2 is the 'middle' point, see Figure 2. Fix indices j, k, l, m. We claim that none of the points $p_{2j}, p_{2jk}, p_{2jkl}, p_{2jklm}$ belong to the 1st quadrant. The following elementary lemma is required.

Lemma 11 *Let points x_1, x_2, x_3 be in the same quadrant on the skyline with x_2 in the middle. Let y be another point such that there exist the following rectangles: (i) R_X containing x_1, x_2, x_3 but not y (ii) R_1 containing the origin and x_1 but not y, (iii) R_3 containing the origin and x_3 but not y, and, (iv) R_y that contains the origin, x_2 and y but not x_1, x_3. Then, y cannot lie in the same quadrant as x_1, x_2, x_3.*

Now, we come to the claim above. To see that p_{2j} cannot belong to the 1st quadrant, we apply Lemma 11 by letting x_1, x_2, x_3 be p_1, p_2, p_3 respectively, y be p_{2j}, and letting $R_X = R, R_1 = R_{11111}, R_3 = R_{31111}, R_y = R_{2jklm}$. (These rectangle choices are not unique; other choices lead to the same conclusion.) Similarly, $p_{2jk}, p_{2jkl}, p_{2jklm}$ belong to different quadrants. By what we showed above, $p_{21}, \ldots, p_{25}$ do not lie in the 1st quadrant. The proof is now essentially successive repetition of the above arguments for the different levels of the tree. For example, applying Lemma 10, we conclude some three of $p_{21}, \ldots, p_{25}$ lie in 1 quadrant - wlog assume this is quadrant 2 and the points are p_{21}, p_{22}, p_{23}, all on the skyline, with p_{22} the middle point. By applying Lemma 11 above we can conclude (wlog) that the points $p_{221}, p_{222}, p_{223}$ must lie in 1 quadrant on the skyline (wlog 3rd quadrant), with p_{222} the middle point. Similarly, applying the combination of Lemma 10 and Lemma 11 to the descendants of p_{222} we conclude as before that all its descendants must lie in quadrant 4. Moreover, all the p_{222l}, for $1 \leq l \leq 5$ must all lie on the skyline and consider a 'middle' point of these - say this is p_{2223}. We have now run out of quadrants for the children at the next level of p_{2223}. More precisely, there is no way to place p_{22231} owing to Lemma 11. This is a contradiction. □

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