Sharing a pizza: bisecting masses with two cuts

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Abstract

Assume you have a pizza consisting of four ingredients (e.g. bread, tomatoes, cheese and olives) that you want to share with your friend. You want to do this fairly, meaning that you and your friend should get the same amount of each ingredient. How many times do you need to cut the pizza so that this is possible? We will show that two straight cuts always suffice. More formally, we will show the following extension of the well-known Ham-sandwich theorem: Given four mass distributions in the plane, they can be simultaneously bisected with two lines. That is, there exist two oriented lines with the following property: let R_1^+ be the region of the plane that lies to the positive side of both lines and let R_2^+ be the region of the plane that lies to the negative side of both lines. Then $R^+ = R_1^+ \cup R_2^+$ contains exactly half of each mass distribution. Additionally, we prove that five mass distributions in $\mathbb{R}^3$ can be simultaneously bisected by two planes.

1 Introduction

The famous *Ham-sandwich theorem* (see e.g. [11, 14]) states that any d mass distributions in $\mathbb{R}^d$ can be simultaneously bisected by a hyperplane. In particular, a two-dimensional sandwich consisting of bread and ham can be cut with one straight cut in such a way that each side of the cut contains exactly half of the bread and half of the ham. However, if two people want to share a pizza, this result will not help them too much, as pizzas generally consist of more than two ingredients. There are two options to overcome this issue: either they don't use a straight cut, but cut along some more complicated curve, or they cut the pizza more than once. In this paper we investigate the latter option. In particular we show that a pizza with four ingredients can always be shared fairly using two straight cuts. See Figure 1 for an example.

To phrase it in mathematical terms, we show that four mass distributions in the plane can be simultaneously bisected with two lines. A precise definition of what bisecting with n lines means is given in the Preliminaries. We further show that five mass distributions in $\mathbb{R}^3$ can be simultaneously bisected by two planes. These two main results are proven in Section 2. In Section 3 we go back to the two-dimensional case and add more restrictions on the lines. In Section 4 we look at the general case of bisecting mass distributions in $\mathbb{R}^d$ with n hyperplanes, and show an upper bound of nd mass distributions that can be simultaneously bisected this way. We conjecture that this bound is tight, that is, that any nd mass distributions in $\mathbb{R}^d$ can be simultaneously bisected with n hyperplanes. For $d = 1$, this is the well-known *Necklace splitting problem*, for which an affirmative answer to our conjecture is known [6, 11]. So, our general problem can be seen as both a generalization of the Ham-sandwich theorem for more than one hyperplane, as well as a generalization of the Necklace splitting problem to higher dimensions.

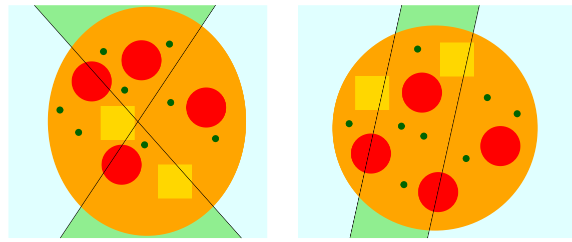

Figure 1: Sharing a (not necessarily round) pizza fairly with two cuts. One person gets the parts in the light blue region, the other person gets the parts in the green region.

Additionally, our results add to a long list of results about partitions of mass distributions, starting with the already mentioned Ham-sandwich theorem. A generalization of this is the polynomial Ham-sandwich theorem, which states that any $\binom{n+d}{d} - 1$ mass distributions in $\mathbb{R}^d$ can be simultaneously bisected by an algebraic surface of degree n [14]. Applied to the problem of sharing a pizza, this result gives an answer on how complicated the cut needs to be, if we want to use only a single (possibly self-intersecting) cut.

Several results are also known about equipartitions of mass distributions into more than two parts. A straightforward application of the 2-dimensional Ham-sandwich theorem is that any mass distribution in the plane can be partitioned into four equal parts with 2 lines. It is also possible to partition a mass distribution in $\mathbb{R}^3$ into 8 equal parts with three planes, but for $d \geq 5$, it is not always possible to partition a mass distribution

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†Partially supported by the ETH Postdoctoral Fellowship

into 2^d equal parts using d hyperplanes [5]. The case $d = 4$ is still open. A result by Buck and Buck [4] states that a mass distribution in the plane can be partitioned into 6 equal parts by 3 lines passing through a common point. Several results are known about equipartitions in the plane with *k-fans*, i.e., k rays emanating from a common point. Note that 3 lines going through a common point can be viewed as a 6-fan, thus the previously mentioned result shows that any mass partition in the plane can be equipartitioned by a 6-fan. Motivated by a question posed by Kaneko and Kano [8], several authors have shown independently that 2 mass distributions in the plane can be simultaneously partitioned into 3 equal parts by a 3-fan [2, 7, 12]. The analogous result for 4-fans holds as well [1]. Partitions into non-equal parts have also been studied [15]. All these results give a very clear description of the sets used for the partitions. If we allow for more freedom, much more is possible. In particular, Soberón [13] and Karasev [9] have recently shown independently that any d mass distributions in $\mathbb{R}^d$ can be simultaneously equipartitioned into k equal parts by k convex sets. The proofs of all of the above mentioned results rely on topological methods, many of them on the famous Borsuk-Ulam theorem and generalizations of it. For a deeper overview of these types of arguments, we refer to Matoušek's excellent book [11].

Preliminaries

A *mass distribution* μ on $\mathbb{R}^d$ is a measure on $\mathbb{R}^d$ such that all open subsets of $\mathbb{R}^d$ are measurable, $0 < \mu(\mathbb{R}^d) < \infty$ and $\mu(S) = 0$ for every lower-dimensional subset S of $\mathbb{R}^d$. Let $\mathcal{L}$ be a set of oriented hyperplanes. For each $\ell \in \mathcal{L}$, let ℓ^+ and ℓ^- denote the positive and negative side of ℓ, respectively (we consider the sign resulting from the evaluation of a point in these sets into the linear equation defining ℓ). For every point $p \in \mathbb{R}^d$, define $\lambda(p) := |\{\ell \in \mathcal{L} \mid p \in \ell^+\}|$ as the number of hyperplanes that have p in their positive side. Let $R^+ := \{p \in \mathbb{R}^d \mid \lambda(p) \text{ is even}\}$ and $R^- := \{p \in \mathbb{R}^d \mid \lambda(p) \text{ is odd}\}$. We say that $\mathcal{L}$ *bisects* a mass distribution μ if $\mu(R^+) = \mu(R^-)$. For a family of mass distributions $\mu_1, \ldots, \mu_k$ we say that $\mathcal{L}$ *simultaneously bisects* $\mu_1, \ldots, \mu_k$ if $\mu_i(R^+) = \mu_i(R^-)$ for all $i \in \{1, \ldots, k\}$.

More intuitively, this definition can also be understood the following way: if C is a cell in the hyperplane arrangement induced by $\mathcal{L}$ and C' is another cell sharing a facet with C, then C is a part of R^+ if and only if C' is a part of R^-. See Figure 2 for an example.

Let $g_i(x) := a_{i,1}x_1 + \ldots + a_{i,d}x_d + a_{i,0} \geq 0$ be the linear equation describing ℓ_i^+ for $\ell_i \in \mathcal{L}$. Then the following is yet another way to describe R^+ and R^-: a point $p \in \mathbb{R}^d$ is in R^+ if $\prod_{\ell_i \in \mathcal{L}} g_i(p) \geq 0$ and it is in R^- if $\prod_{\ell_i \in \mathcal{L}} g_i(p) \leq 0$. That is, if we consider the union of the hyperplanes in $\mathcal{L}$ as an oriented algebraic surface of degree $|\mathcal{L}|$, then R^+ is the positive side of this surface and R^- is the negative side.

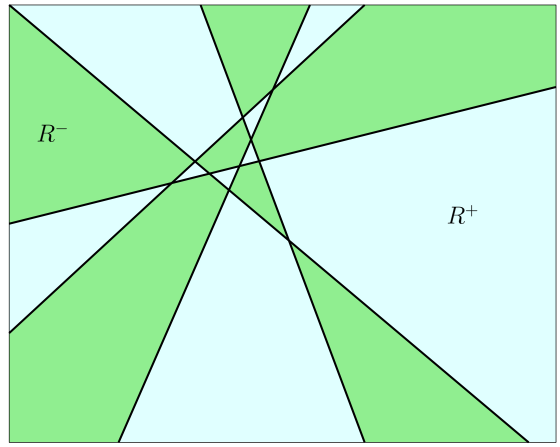

Figure 2: The regions R^+ (light blue) and R^- (green).

Note that reorienting one line just maps R^+ to R^- and vice versa. In particular, if a set $\mathcal{L}$ of oriented hyperplanes simultaneously bisects a family of mass distributions $\mu_1, \ldots, \mu_k$, then so does any set $\mathcal{L}'$ of the same hyperplanes with possibly different orientations. Thus we can ignore the orientations and say that a set $\mathcal{L}$ of (undirected) hyperplanes simultaneously bisects a family of mass distributions if some orientation of the hyperplanes does.

2 Two Cuts

In this section we will look at simultaneous bisections with two lines in $\mathbb{R}^2$ and with two planes in $\mathbb{R}^3$. Both proofs rely on the famous Borsuk-Ulam theorem [3], which we will use in the version of *antipodal mappings*. An antipodal mapping is a continuous mapping $f : S^d \to \mathbb{R}^d$ such that $f(-x) = -f(x)$ for all $x \in S^d$.

Theorem 1 (Borsuk-Ulam theorem [11]) *For every antipodal mapping $f : S^d \to \mathbb{R}^d$ there exists a point $x \in S^d$ satisfying $f(x) = 0$.*

The proof of the Ham-sandwich theorem can be derived from the Borsuk-Ulam theorem in the following way. Let μ_1 and μ_2 be two mass distributions in $\mathbb{R}^2$. For a point $p = (a, b, c) \in S^3$, consider the equation of the line $ax + by + c = 0$ and note that it defines a line in the plane parametrized by the coordinates of p. Moreover, it splits the plane into two regions, the set $R^+(p) = \{(x, y) \in \mathbb{R}^2 : ax + by + c \geq 0\}$ and the set $R^-(p) = \{(x, y) \in \mathbb{R}^2 : ax + by + c \leq 0\}$. Thus, we can define two functions $f_i := \mu_i(R^+(p)) - \mu_i(R^-(p))$ that together yield a function $f : S^2 \to \mathbb{R}^2$ that is continuous and antipodal. Thus, by the Borsuk-Ulam theorem, there is a point $p = (a, b, c) \in S^2$, such that

$f_i(-p) = -f_i(p)$ for $i \in \{1, 2\}$, which implies that the line $ax+by+c = 0$ defined by p is a Ham-sandwich cut. In this paper, we use variants of this proof idea to obtain simultaneous bisections by geometric objects that are parametrized by points in S^d. The main difference is that we replace some of the f_i's by other functions, whose vanishing enforces specific structural properties on the resulting bisecting object. We are now ready to prove our first main result:

Theorem 2 *Let $\mu_1, \mu_2, \mu_3, \mu_4$ be four mass distributions in $\mathbb{R}^2$. Then there exist two lines ℓ_1, ℓ_2 such that $\{\ell_1, \ell_2\}$ simultaneously bisects $\mu_1, \mu_2, \mu_3, \mu_4$.*

Proof. For each $p = (a, b, c, d, e, g) \in S^5$ consider the bivariate polynomial $c(p)(x, y) = ax^2+by^2+cxy+dx+ey+g$. Note that $c(p)(x, y) = 0$ defines a conic section in the plane. Let $R^+(p) := \{(x, y) \in \mathbb{R}^2 \mid c(p)(x, y) \geq 0\}$ be the set of points that lie on the positive side of the conic section and let $R^-(p) := \{(x, y) \in \mathbb{R}^2 \mid c(p)(x, y) \leq 0\}$ be the set of points that lie on its negative side. Note that for $p = (0, 0, 0, 0, 0, 1)$ we have $R^+(p) = \mathbb{R}^2$ and $R^-(p) = \emptyset$, and vice versa for $p = (0, 0, 0, 0, 0, -1)$. Also note that $R^+(-p) = R^-(p)$. We now define four functions $f_i : S^5 \to \mathbb{R}$ as follows: for each $i \in \{1, \ldots, 4\}$ define $f_i := \mu_i(R^+(p)) - \mu_i(R^-(p))$. From the previous observation it follows immediately that $f_i(-p) = -f_i(p)$ for all $i \in \{1, \ldots, 4\}$ and $p \in S^5$. It can also be shown that the functions are continuous, but for the sake of readability we postpone this step to the end of the proof. Further let

$$A(p) := \det \begin{pmatrix} a & c/2 & d/2 \\ c/2 & b & e/2 \\ d/2 & e/2 & g \end{pmatrix}.$$

It is well-known that the conic section $c(p)(x, y) = 0$ is degenerate if and only if $A(p) = 0$. Furthermore, being a determinant of a 3×3-matrix, A is continuous and $A(-p) = -A(p)$. Hence, setting $f_5(p) := A(p)$, $f := (f_1, \ldots, f_5)$ is an antipodal mapping from S^5 to $\mathbb{R}^5$, and thus by the Borsuk-Ulam theorem, there exists p^* such that $f(p^*) = 0$. For each $i \in \{1, \ldots, 4\}$ the condition $f_i(p^*) = 0$ implies by definition that $\mu_i(R^+(p^*)) = \mu_i(R^-(p^*))$. The condition $f_5(p^*) = 0$ implies that $c(p)(x, y) = 0$ describes a degenerate conic section, i.e., two lines, a single line of multiplicity 2, a single point or the empty set. For the latter three cases, we would have $R^+(p^*) = \mathbb{R}^2$ and $R^-(p^*) = \emptyset$ or vice versa, which would contradict $\mu_i(R^+(p^*)) = \mu_i(R^-(p^*))$. Thus $f(p^*) = 0$ implies that $c(p)(x, y) = 0$ indeed describes two lines that simultaneously bisect $\mu_1, \mu_2, \mu_3, \mu_4$.

It remains to show that f_i, is continuous for $i \in \{1, \ldots, 4\}$. To that end, we will show that $\mu_i(R^+(p))$ is continuous. The same arguments apply to $\mu_i(R^-(p))$, which then shows that f_i being the difference of two continuous functions is continuous. So let $(p_n)_{n=1}^{\infty}$ be a sequence of points in S^5 converging to p. We need to show that $\mu_i(R^+(p_n))$ converges to $\mu_i(R^+(p))$. If a point q is not on the boundary of $R^+(p)$, then for all n large enough we have $q \in R^+(p_n)$ if and only if $q \in R^+(p)$. As the boundary of $R^+(p)$ has dimension 1 and μ_i is a mass distribution we have $\mu_i(\partial R^+(p)) = 0$ and thus $\mu_i(R^+(p_n))$ converges to $\mu_i(R^+(p))$ as required. ☐

Using similar ideas, we can also prove a result in $\mathbb{R}^3$. For this we first need the following lemma:

Lemma 3 *Let $h(x, y, z)$ be a quadratic polynomial in 3 variables. Then there are antipodal functions $g_1, \ldots, g_4$, each from the space of coefficients of h to $\mathbb{R}$, whose simultaneous vanishing implies that h factors into linear polynomials.*

Proof. Write h as

$$h = (x, y, z, 1) \cdot A \cdot (x, y, z, 1)^T,$$

where A is a 4×4-matrix depending on the coefficients of h. It is well-known that h factors into linear polynomials if and only if the rank of A is at most 2. A well-known sufficient condition for this is that the determinants of all (3×3)-minors of A vanish. There are $\binom{4}{3} = 4$ different (3×3)-minors and for each of them the determinant is an antipodal function. ☐

With this, we can now prove the following:

Theorem 4 *Let $\mu_1, \ldots, \mu_5$ be five mass distributions in $\mathbb{R}^3$. Then there exist two planes ℓ_1, ℓ_2 such that $\{\ell_1, \ell_2\}$ simultaneously bisects $\mu_1, \ldots, \mu_5$.*

Proof. Similar to the proof of Theorem 2, we map a point $p \in S^9$ to a quadratic polynomial $h(p)(x, y, z)$ (note that a quadratic polynomial in three variables has 10 coefficients). Let $R^+(p) := \{(x, y, z) \in \mathbb{R}^2 \mid h(p)(x, y, z) \geq 0\}$ be the set of points that lie on the positive side of the conic section and let $R^-(p) := \{(x, y, z) \in \mathbb{R}^3 \mid h(p)(x, y) \leq 0\}$ be the set of points that lie on the negative side. For each $i \in \{1, \ldots, 5\}$ define $f_i := \mu_i(R^+(p)) - \mu_i(R^-(p))$. Analogous to the proof of Theorem 2, these functions are continuous and $f_i(-p) = -f_i(p)$. Further let $g_1, \ldots, g_4$ be the four functions constructed in Lemma 3. Then $f := (f_1, \ldots, f_5, g_1, \ldots, g_4)$ is a continuous antipodal mapping from S^9 to $\mathbb{R}^9$. Thus, by the Borsuk-Ulam theorem there exists a point $p^* \in S^9$ such that $f(p^*) = 0$. Analogous to the proof of Theorem 2, the existence of such a point implies the claimed result. ☐

3 Putting more restrictions on the cuts

In this section, we look again at bisections with two lines in the plane. However, we enforce additional conditions on the lines, at the expense of being only able to simultaneously bisect fewer mass distributions.

Theorem 5 *Let μ_1, μ_2, μ_3 be three mass distributions in $\mathbb{R}^2$. Given any line ℓ in the plane, there exist two lines ℓ_1, ℓ_2 such that $\{\ell_1, \ell_2\}$ simultaneously bisects μ_1, μ_2, μ_3 and ℓ_1 is parallel to ℓ.*

Proof. Assume without loss of generality that ℓ is parallel to the x-axis; otherwise rotate μ_1, μ_2, μ_3 and ℓ to achieve this property. Consider the conic section defined by the polynomial $ax^2+by^2+cxy+dx+ey+g$. If $a=0$ and the polynomial decomposes into linear factors, then one of the factors must be of the form $\beta y+\gamma$. In particular, the line defined by this factor is parallel to the x-axis. Thus, we can modify the proof of Theorem 2 in the following way: we define f_1, f_2, f_3 and f_5 as before, but set $f_4 := a$. It is clear that f still is an antipodal mapping. The zero of this mapping now implies the existence of two lines simultaneously bisecting three mass distributions, one of them being parallel to the x-axis, which proves the result. □

Another natural condition on a line is that it has to pass through a given point.

Theorem 6 *Let μ_1, μ_2, μ_3 be three mass distributions in $\mathbb{R}^2$ and let q be a point. Then there exist two lines ℓ_1, ℓ_2 such that $\{\ell_1, \ell_2\}$ simultaneously bisects μ_1, μ_2, μ_3 and ℓ_1 goes through q.*

Proof. Assume without loss of generality that q coincides with the origin; otherwise translate μ_1, μ_2, μ_3 and q to achieve this. Consider the conic section defined by the polynomial $ax^2+by^2+cxy+dx+ey+g$. If $g=0$ and the polynomial decomposes into linear factors, then one of the factors must be of the form $\alpha x+\beta y$. In particular, the line defined by this factor goes through the origin. Thus, we can modify the proof of Theorem 2 in the following way: we define f_1, f_2, f_3 and f_5 as before, but set $f_4 := g$. It is clear that f still is an antipodal mapping. The zero of this mapping now implies the existence of two lines simultaneously bisecting three mass distributions, one of them going through the origin, which proves the result. □

We can also enforce the intersection of the two lines to be at a given point, but at the cost of another mass distribution.

Theorem 7 *Let μ_1, μ_2 be two mass distributions in $\mathbb{R}^2$ and let q be a point. Then there exist two lines ℓ_1, ℓ_2 such that $\{\ell_1, \ell_2\}$ simultaneously bisects μ_1, μ_2, and both ℓ_1 and ℓ_2 go through q.*

Proof. Assume without loss of generality that q coincides with the origin; otherwise translate μ_1, μ_2 and q to achieve this. Consider the conic section defined by the polynomial ax^2+by^2+cxy, i.e., the conic section where $d=e=g=0$. If this conic section decomposes into linear factors, both of them must be of the form $\alpha x+\beta y=0$. In particular, both of them pass through the origin. Furthermore, as $d=e=g=0$, the determinant $A(p)$ vanishes, which implies that the conic section is degenerate. Thus, we can modify the proof of Theorem 2 in the following way: we define f_1, f_2 as before, but set $f_3 := d$, $f_4 := e$ and $f_5 := g$. It is clear that f still is an antipodal mapping. The zero of this mapping now implies the existence of two lines simultaneously bisecting two mass distributions, both of them going through the origin, which proves the result. □

4 The general case

In this section we consider the more general question of how many mass distributions can be simultaneously bisected by n hyperplanes in $\mathbb{R}^d$. We introduce the following conjecture:

Conjecture 1 *Any $n \cdot d$ mass distributions in $\mathbb{R}^d$ can be simultaneously bisected by n hyperplanes.*

For $n=1$ this is equivalent to the Ham-sandwich theorem. Theorem 2 proves this conjecture for the case $d=n=2$. We first observe that the number of mass distributions would be tight:

Observation 1 *There exists a family of $n \cdot d+1$ mass distributions in $\mathbb{R}^d$ that cannot be simultaneously bisected by n hyperplanes.*

Proof. Let $P=\{p_1, \ldots, p_{nd+1}\}$ be a finite point set in $\mathbb{R}^d$ in general position (no $d+1$ of them on the same hyperplane). Let ϵ be the smallest distance of a point to a hyperplane defined by d other points. For each $i \in \{1, \ldots, nd+1\}$ define μ_i as the volume measure of $B_i := B_{p_i}(\frac{\epsilon}{2})$. Note that any hyperplane intersects at most d of the B_i's. On the other hand, for a family of n hyperplanes to bisect μ_i, at least one of them has to intersect B_i. Thus, as n hyperplanes can intersect at most $n \cdot d$ different B_i's, there is always at least one μ_i that is not bisected. □

A possible way to prove the conjecture would be to generalize the approach from Section 2 as follows: Consider the n hyperplanes as a highly degenerate algebraic surface of degree n, i.e., the zero set of a polynomial of degree n in d variables. Such a polynomial has $k := \binom{n+d}{d}$ coefficients and can thus be seen as a point on S^{k-1}. In particular, we can define $\binom{n+d}{d}-1$ antipodal mappings to $\mathbb{R}$ if we want to apply the Borsuk-Ulam theorem. Using $n \cdot d$ of them to enforce the mass distributions to be bisected, we can still afford $\binom{n+d}{d}-nd-1$ antipodal mappings to enforce the required degeneracies of the surface. There are many conditions known to enforce such degeneracies, but they all require far too many mappings or use mappings that are not antipodal. Nonetheless the following conjecture implies Conjecture 1:

Conjecture 2 *Let C be the space of coefficients of polynomials of degree n in d variables. Then there exists a family of $\binom{n+d}{d} - nd - 1$ antipodal mappings $g_i : C \to \mathbb{R}$, $i \in \{1, \dots, \binom{n+d}{d} - nd - 1\}$ such that $g_i(c) = 0$ for all i implies that the polynomial defined by the coefficients c decomposes into linear factors.*

5 Algorithmic remarks

Going back to the planar case, instead of considering four mass distributions $\mu_1, \dots, \mu_4$, one can think of having four sets of points $P_1, \dots, P_4 \subset \mathbb{R}^2$. Thus, our problem translates to finding two lines that simultaneously bisect these point sets. The existence of such a bisection follows Theorem 2 as we can always replace each point by a sufficiently small disk and consider their area as a mass distribution.

An interesting question is then to find efficient algorithms to compute such a bisection given any four sets $P_1, \dots, P_4$ with a total of n points. It is known that linear time algorithms exist for Ham-sandwich cuts of two sets of points in $\mathbb{R}^2$. However, we have not been able to obtain similar results for simultaneous bisections using two lines. A trivial $O(n^5)$ time algorithm can be applied by looking at all pairs of combinatorially different lines. While this running time can be reduced using known data structures, it still goes through $\Theta(n^4)$ different pairs of lines. Finding a better algorithm remains an interesting open question.

Using the algorithms for Ham-sandwich cuts from Lo, Steiger and Matoušek [10], and a Veronese map one can compute a conic section that simultaneously bisects $P_1, \dots, P_4$ in $O(n^{4-\varepsilon})$ time. It remains open whether this algorithm can be modified to use the last degree of freedom to guarantee the degeneracy of the conic section and achieve $o(n^4)$ time.

References

[1] I. Bárány and J. Matoušek. Equipartition of two measures by a 4-fan. *Discrete & Computational Geometry*, 27(3):293–301, 2002.

[2] S. Bespamyatnikh, D. Kirkpatrick, and J. Snoeyink. Generalizing Ham Sandwich Cuts to Equitable Subdivisions. *Discrete & Computational Geometry*, 24(4):605–622, 2000.

[3] K. Borsuk. Drei Sätze ber die n-dimensionale euklidische Sphäre. *Fundamenta Mathematicae*, 20(1):177–190, 1933.

[4] R. C. Buck and E. F. Buck. Equipartition of convex sets. *Mathematics Magazine*, 22(4):195–198, 1949.

[5] H. Edelsbrunner. *Algorithms in Combinatorial Geometry.* Springer-Verlag New York, Inc., New York, NY, USA, 1987.

[6] C. R. Hobby and J. R. Rice. A moment problem in L_1 approximation. *Proceedings of the American Mathematical Society*, 16(4):665–670, 1965.

[7] H. Ito, H. Uehara, and M. Yokoyama. 2-dimension ham sandwich theorem for partitioning into three convex pieces. In *Revised Papers from the Japanese Conference on Discrete and Computational Geometry*, JCDCG '98, pages 129–157, London, UK, 2000. Springer-Verlag.

[8] A. Kaneko and M. Kano. Balanced partitions of two sets of points in the plane. *Computational Geometry*, 13(4):253 – 261, 1999.

[9] R. N. Karasev. Equipartition of several measures. *ArXiv e-prints*, 1011.4762, Nov. 2010.

[10] C.-Y. Lo, J. Matoušek, and W. Steiger. Algorithms for ham-sandwich cuts. *Discrete & Computational Geometry*, 11(1):433–452, 1994.

[11] J. Matoušek. *Using the Borsuk-Ulam Theorem: Lectures on Topological Methods in Combinatorics and Geometry.* Springer Publishing Company, Incorporated, 2007.

[12] T. Sakai. Balanced convex partitions of measures in $\mathbb{R}^2$. *Graphs and Combinatorics*, 18(1):169–192, 2002.

[13] P. Soberón. Balanced convex partitions of measures in $\mathbb{R}^d$. *Mathematika*, 58(01):71–76, 2012.

[14] A. H. Stone and J. W. Tukey. Generalized "sandwich" theorems. *Duke Math. J.*, 9(2):356–359, 06 1942.

[15] R. T. Zivaljevic. Combinatorics and Topology of partitions of spherical measures by 2 and 3 fans. *ArXiv e-prints*, math/0203028, Mar. 2002.