Abstract

Last year, a new notion of rep-cube was proposed. A rep-cube is a polyomino that is a net of a cube, and it can be divided into some polyominoes such that each of them can be folded to a cube. This notion was inspired by the notions of polyomino and rep-tile, which were introduced by Solomon W. Golomb. It was proved that there are infinitely many distinct rep-cubes. In this paper, we investigate this new notion and obtain three new results. First, we prove that there does not exist a regular rep-cube of order 3, which solves an open question proposed in the paper. Next, we enumerate all regular rep-cubes of order 2 and 4. For example, there are 33 rep-cubes of order 2; that is, there are 33 dodecominoes that can fold to a cube of size $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$ and each of them can be divided into two nets of unit cube. Similarly, there are 7185 rep-cubes of order 4. Lastly, we focus on pythagorean triples that consist of three positive integers $(a, b, c)$ with $a^2 + b^2 = c^2$. For each of these triples, we can consider a rep-cube problem that asks whether a net of a cube of size $c \times c \times c$ can be divided into two nets of two cubes of size $a \times a \times a$ and $b \times b \times b$. We give a partial answer to this natural open question by dividing into more than two pieces. For any given pythagorean triple $(a, b, c)$, we construct five polyominoes that form a net of a cube of size $c \times c \times c$ and two nets of two cubes of size $a \times a \times a$ and $b \times b \times b$.

1 Introduction

A polyomino is a “simply connected” set of unit squares introduced by Solomon W. Golomb in 1954 [7]. Since then, polyominoes have been playing an important role in recreational mathematics (see, e.g., [5]). In 1962, Golomb also proposed an interesting notion called “rep-tile”: a polygon is a rep-tile of order $k$ if it can be divided into $k$ replicas congruent to one another and similar to the original (see [6, Chap 19]).

From these notions, Abel et al. proposed a new notion [1]; a polyomino is said to be a rep-cube of order $k$ if it is a net of a cube (or, it can fold to a cube), and it can be divided into $k$ polyominoes such that each of them can fold to a cube. If all $k$ polyominoes have the same size, we call the original polyomino a regular rep-cube of order $k$. We note that crease lines are not necessarily along the edges of the polyomino. For example, a regular rep-cube of order 2 folds to a cube by folding along the diagonals of unit squares; see Figure 1.

In [1], Abel et al. propose regular rep-cubes of order $k$ for each $k = 2, 4, 5, 8, 9, 36, 50, 64$, and also $k = 36gk^2$ for any positive integer $k'$ and an integer $g$ in \{2, 4, 5, 8, 9, 36, 50, 64\}. In other words, there are infinitely many $k$ that allow regular rep-cube of order $k$. On the other hand, they left an open problem that asks if there is a rep-cube of order 3. In this paper, we first answer to this question. There are no regular rep-cube of order 3.

Next we enumerate all possible regular rep-cubes of order $k$ for small $k$. We mention that the following problem is not so easy to solve efficiently; for a given polygon $P$, determine if $P$ can fold to a cube or not. Recently, Horiyama and Mizunashi developed an efficient algorithm that solves this problem for a given orthogonal polygon, which runs in $O((n+m) \log n)$ time, where $n$ is the number of vertices in $P$, and $m$ is the maximum number of line segments that appears on a crease line [8]. We remark that the parameter $m$ is hidden and can be huge comparing to $n$. In our case, $P$ is a polyomino, and this hidden parameter is linear to the number of unit squares in $P$, and hence our algorithm is simpler.

Finally, we investigate non-regular rep-cube. In [1], Abel et al. also asked if there exists a rep-cube of area 150 that is a net of a cube of size $5 \times 5 \times 5$ and it can be divided into two nets of cubes of size $3 \times 3 \times 3$ and $4 \times 4 \times 4$. This idea comes from a pythagorean triple $(3, 4, 5)$ with $3^2 + 4^2 = 5^2$. We give a partial answer to this question by dividing into more pieces than 2. We give a general way for any pythagorean triple $(a, b, c)$ with $a < b < c$ to obtain five piece solution. That is, for any given pythagorean triple $(a, b, c)$ with $a < b < c$, we construct a polyomino that is a net of a cube of $c \times c \times c$, and it can be divided into 5 pieces such that one of 5 pieces can fold to a cube of $a \times a \times a$, and gluing the
remaining 4 pieces, we can obtain a net of a cube of \( b \times b \times b \).

Due to lack of space, some proofs are omitted.

2 Nonexistence of regular rep-cubes

The main theorem in this section is the following.

**Theorem 1** There does not exist a regular rep-cube of order 3.

We first show two lemmas (proofs are omitted):

**Lemma 2** Let \( Q \) be a cube and \( P \) any development\(^1\) of \( Q \). Then \( P \) is concave.

Let \( P \) be a polyomino (not necessarily hexomino) that can fold to a cube \( Q \). Then, by Lemma 2, \( P \) has no “rolling bell” (see [4] for further details). This fact implies that, when we fold \( P \) to \( Q \), each vertex on \( Q \) should appear at either the grid point of \( P \) or the middle point of a unit edge in \( P \). For these vertices of \( Q \), we state stronger property:

**Lemma 3** Let \( P \) be a polyomino that can fold to a cube \( Q \). Let \( \ell \) be the length of an edge of \( Q \). (That is, \( P \) is a \( 6\ell^2 \)-omino.) Then \( P \) can be placed on a grid of size \( \ell \) so that every vertex of \( Q \) on \( P \) is on a grid point. I.e., not only all vertices on \( Q \) appear on the boundary of \( P \), but also they are also aligned on the grid points of size \( \ell \).

Now we turn to the proof of Theorem 1. We assume that there exists a regular rep-cube of order 3, and derive contradictions. That is, we assume that there is a polyomino \( \hat{P} \) such that \( \hat{P} \) can be divided into three polyominoes \( P_1, P_2, P_3 \) of the same size, and each of \( P, P_1, P_2, P_3 \) can fold to a cube of certain size. Let \( \overrightarrow{Q} \) and \( \overrightarrow{S} \) denote the cubes folded from \( P \) and \( P_1 \), respectively. We suppose that the length of an edge of \( Q \) is \( \ell \). That is, \( P_1 \) is a \( 6\ell^2 \)-omino, and \( P \) is a \( 18\ell^2 \)-omino. We remark that \( \ell \) is not necessarily an integer, but \( 6\ell^2 \) is.

Now we consider the polyomino \( P_1 \); that is a \( 6\ell^2 \)-omino, and folds to the cube \( Q_1 \) of size \( \ell \times \ell \times \ell \). Then, by Lemma 3, \( P_1 \) can be on the grid of size \( \ell \) so that every vertex of \( Q_1 \) is on the grid. We take any two vertices \( v_1 \) and \( v_2 \) of \( Q_1 \) of distance \( \ell \) on the grid. Then the vector \( \overrightarrow{v_1v_2} \) can be represented by \( (a, b) \) for some nonnegative integers \( a \) and \( b \). That is, \( a^2 + b^2 = \ell^2 \) for some integers \( a \) and \( b \). (The same idea can be found in [4, Ch. 5.1.1] and [3].) We can apply the same argument to \( P \) and \( Q \), and hence there are some nonnegative integers \( \hat{a} \) and \( \hat{b} \) such that \( \hat{a}^2 + \hat{b}^2 = 3\ell^2 \). Thus we obtain \( \hat{a}^2 + \hat{b}^2 = 3(a^2 + b^2) \).

Therefore, it is sufficient to show that there are no such integers. To derive a contradiction, we assume that we have \( \hat{a}^2 + \hat{b}^2 = 3(\hat{a}^2 + \hat{b}^2) \), and they are the minimum integers with respect to the value of \( \hat{a}^2 + \hat{b}^2 \).

Now, for an integer \( i \), \((3i \pm 1)^2 = 9i^2 \pm 6i + 1 \). Therefore, a square number \( x \) is either \( x = 3x' \) or \( 3x' + 1 \) for some integer \( x' \). Since \( \hat{a}^2 + \hat{b}^2 = 3(a^2 + b^2) \) is a multiple of 3, both of \( \hat{a} \) and \( \hat{b} \) are multiples of 3, say \( \hat{a} = 3\hat{a}' \) and \( \hat{b} = 3\hat{b}' \). Then, we have \((3\hat{a}')^2 + (3\hat{b}')^2 = 9(\hat{a}'^2 + \hat{b}'^2) = 3(a^2 + b^2) \). Thus we obtain \( a^2 + b^2 = 3(\hat{a}'^2 + \hat{b}'^2) \). This contradicts the minimality of the value of \( \hat{a}^2 + \hat{b}^2 \). Therefore, we have no such integers \( a, b, \hat{a}, \hat{b} \). This completes the proof of Theorem 1. \( \square \)

3 Enumeration of regular rep-cubes

In this section, we describe the exhaustive search algorithm for generating all regular rep-cubes of order \( k \) (for \( k = 2 \) and \( k = 4 \)).

Algorithm 1 gives the outline of this algorithm. It works as follows: Let \( S_1 \) be the set of all \((6 \times i)\)-ominoes such that (1) it is composed by \( i \) nets of a unit cube, (2) it can cover a part of a cube of size \( \sqrt{k} \times \sqrt{k} \times \sqrt{k} \). In the term of search of development, each element in \( S_1 \) is called a partial development of a cube of size \( \sqrt{k} \times \sqrt{k} \times \sqrt{k} \) [10]. That is, \( S_1 \) is the set of all nets of a unit cube, which consists of 11 hexominoes, and each set \( S_i \) with \( i > 1 \) is a subset of \((6 \times i)\)-ominoes that can be computed from \( S_{i−1} \). Let \( P_1 \) be any polyomino in \( S_1 \), e.g., \( P_1 \) is one of the 11 hexominoes in \( S_1 \).

In Procedure CheckCover, the algorithm checks if \( P_1 \) can cover the cube of size \( \sqrt{k} \times \sqrt{k} \times \sqrt{k} \) without overlap. The details will be described later. Our final goal is to obtain the set \( S_k \) that contains all regular rep-cubes of order \( k \) from the set \( S_1 \).

**Algorithm 1:** Outline of the exhaustive search algorithm.

**Input:** Integer \( k \) of the order for the rep-cube;  
**Output:** All rep-cubes in \( S_k \);  
for \( i = 2 \) to \( k \) do

1. **foreach** partial development \( P_{i−1} \) in \( S_{i−1} \) do

2. **foreach** development \( P_i \) in \( S_i \) do

3. attach \( P_i \) to \( P_{i−1} \) at each possible adjacency square on the boundary of \( P_{i−1} \) to obtain a new polyomino \( P_i \);

4. if CheckCover(\( P_i \)) \( == \) 1 then

5. store \( P_i \) into \( S_i \); // \( P_i \) is a partial

6. // development of the box of

7. \( = \) \( \sqrt{k} \times \sqrt{k} \times \sqrt{k} \)

9. return \( S_k \);

The algorithm works in a loop as follows. It picks up a polyomino \( P_{i−1} \) in \( S_{i−1} \) and a hexomino \( P_i \) in

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\(^1\)We use “net” that has no overlap when it is spread out. When we use “development,” overlap is not yet considered. 
lines of the cube of $P$ generates a new polyomino net. boundary of a squares on the possible adjacency relationship. Therefore, two unit squares share an edge on $P$. That is, two unit squares share an edge on $Q$, folded from a polyomino $P$. Then we can obtain a part of the same adjacency relationship. Therefore, we can decide if a polyomino $P$ can fold to a cube $Q$ by checking the positional relationship of the unit squares in Procedure CheckCover.

As mentioned in Introduction, we find that the folding lines of the cube of $\sqrt{k} \times \sqrt{k} \times \sqrt{k}$ are not along the edges of unit squares. Since the rep-cubes of order 2 and 4 have different folding ways, we need a universal method to check whether a polyomino is a partial development or not. In [10], the authors proposed an algorithm that checks the positional relationships of unit squares on the polyomino. Consider any polyhedron, e.g., a cube $Q$, folded from a polyomino $P$. Then we can obtain an adjacency relationship of unit squares in $P$ on $Q$. That is, two unit squares share an edge on $P$ only if they share it on $Q$. Thus any development of $Q$ keeps a part of the same adjacency relationship. Therefore, we can decide if a polyomino $P$ can fold to a cube $Q$ by checking the positional relationship of the unit squares in Procedure CheckCover.

We consider the first development in Figure 5 as an example $P$ (Figure 3). We first mark a unit square with the number 1 as the start point. Then we mark all of its neighbors a number according to the adjacency list of cube of size $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$ as in Table 1 and Figure 4 in all four directions. (For example, the square 1 is surrounded by 12(above), 11(right), 2(below), and 3(left) from the viewpoint of the square 1.) This step is extended to its farther neighbors until every square of $P$ is marked with a number. After this step, if every square in connected $P$ is marked with its unique number, $P$ can wrap the cube of size $\sqrt{k} \times \sqrt{k} \times \sqrt{k}$ with consistency. On the other hand, if (1) one square is marked with different numbers by its neighbors or (2) two or more squares are marked with the same number, then an overlap occurs in this folding way of $P$. We check all possible start points and directions for each $P$.

As a result of finding the rep-cube of order 2, by putting two developments of a cube aside, there are 2424 distinct dodecominoes. Among them, there are 33 regular rep-cubes of order 2 that can fold to a cube of size $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$ and each of them can be divided into two nets of a unit cube. As shown in Figures 5 and 6, we can observe that 17 rep-cubes out of 33 consist of two nets of the same shape. We call them uniform rep-cubes. Precisely, we say a regular rep-cube of order $k$ is uniform if its all $k$ nets are the same shape.
For the case of finding the regular rep-cube of order 4, we also implement this algorithm. As a result, we got the amount of partial developments of $i$ pieces as in Table 2, which means there are 7185 regular rep-cubes of order 4. Among them, we also find all uniform rep-cubes of order 4, which are 158 in total. One example of these uniform rep-cubes is shown in Figure 7. Out of 158, 98 of these uniform rep-cubes are made of pieces in shape (b) shown in Figure 8.

Table 2: The number of partial developments of regular rep-cubes of order 4.

<table>
<thead>
<tr>
<th>Set of partial developments</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number of developments</td>
<td>11</td>
<td>2345</td>
<td>114852</td>
<td>7185</td>
</tr>
</tbody>
</table>

In Figure 1 of [1], they gave three uniform rep-cubes of order 2 (Figure 1), 4, and 9. On the other hand, in [1], they also show a regular rep-cube of order 50 that contains all kinds of 11 nets of a unit cube. It may worth focusing on these special cases for a larger $k$.

In the analysis of the results, we found two different patterns of shapes that can make the same rep-cube. As shown in Figure 9, except for the difference in composition, these two rep-cubes have the same contour, the same surface area and the same folding way. Finding this kind of rep-cube can be an interesting topic in the future research.

Figure 5: All 17 uniform rep-cubes of order 2.

Figure 6: All regular rep-cubes of order 2 that are not uniform.

Figure 7: An example of uniform rep-cubes of order 4.

Figure 8: List of the amount of uniform rep-cubes of order 4 made by each of 11 shapes.
4 Rep-cubes based on pythagorean triples

A pythagorean triple is a 3-tuple of positive integers that satisfies \(a^2 + b^2 = c^2\). In [1], Abel et al. propose an interesting open question related to the pythagorean triple. That is, the question asks whether there is a rep-cube of order 2 of area \(c^2\) such that (1) it folds to a \(c \times c \times c\) cube, and (2) it can be divided into two polyominoes so that they fold to a \(a \times a \times a\) cube and another \(b \times b \times b\) cube. The most famous one is \((3, 4, 5)\) with \(3^2 + 4^2 = 5^2\). We note that for any pythagorean triple \((a, b, c)\), for any positive integer \(k\), \((ak, bk, ck)\) is also a pythagorean triple. However, we only consider pythagorean triples with \(GCD(a, b, c) = 1\). Then, it is known that a triple \((a, b, c)\) with \(GCD(a, b, c) = 1\) is a pythagorean triple if and only if there are two positive integers \(m, n\) such that (1) \(m - n\) is odd, and we can obtain a pythagorean triple as \((m^2 - n^2, 2mn, m^2 + n^2)\) for these \(n\) and \(m\).

It is trivial that when we divide any net of a \(c \times c \times c\) cube into \(6c^2\) unit squares, we can make two cubes of size \(a \times a \times a\) and \(b \times b \times b\). Therefore, we can consider this open problem as an optimization problem to minimize the number of polyominoes that can form both of a net of \(c \times c \times c\) cube, and two nets of two cubes of size \(a \times a \times a\) and \(b \times b \times b\). In this section, we give the following theorem:

**Theorem 4** Let \((a, b, c)\) be any pythagorean triple with \(a < b < c\). Then we can construct a set \(S(a, b, c)\) of five polyominoes such that (1) the polyominoes can form a net of \(c \times c \times c\) cube, and (2) they can form two nets of two cubes of size \(a \times a \times a\) and \(b \times b \times b\).

We here show an example in Figure 10 to get the idea. When we choose a pythagorean triple \((3, 4, 5)\), the polyomino in Figure 10(a) folds to a \(3 \times 3 \times 3\) cube, and the polyomino in Figure 10(b) folds to a \(4 \times 4 \times 4\) cube. It is less intuitive, however, the reader can obtain a \(5 \times 5 \times 5\) cube from the polyomino in Figure 10(c) by folding along the dotted lines. Here we give a general construction for any pythagorean triple.

**Proof.** We first give a brief idea of the construction in Figure 11. The first step is that we open two small cubes of size \(a \times a \times a\) and \(b \times b \times b\) at their any vertices. We cut along the three lines from the vertex until we have a kind of a triangular-pyramid-like shape; each rectangular face consists of two squares, and these three rectangles are glued like in wind-wheel shape. Then we regard these two triangular pyramids as cone-like shapes, and attach each of apexes to the two opposite vertices of the big cube of size \(c \times c \times c\). That is, they are glued to two endpoints of a diagonal of the big cube.

The main trick is that the grids of two small cubes are not aligned to the grid of the big cube; we twist two cones so that their edges (or grid lines) make two edges of pythagorean triangle of length \(a\) and \(b\). As a result, three vertices of the big cube are on the boundary of the wind-wheel shape made from the cube of size \(b \times b \times b\), and the other three vertices of the big cube are on the boundary of the other wind-wheel shape made from the cube of size \(a \times a \times a\). Then, we have two cases depending on the size of these two small cubes.

The first case is that \(a < b < 2a\). For example, the most famous pythagorean triple \((3, 4, 5)\) (for \(m = 2, n = 1\)) satisfies this condition. In this case, the situation is illustrated on the net of the big cube in Figure 12. The outline is the net of the big cube, and three vertices labeled by \(p\) form a vertex of the big cube, and the apex of the cone made by the small cube of size \(a \times a \times a\)
We now let $|yz| = c$ to the right triangle $a|y| = a$, $|yz| = b$, and $|zx| = c$. We now let $a' = b - a$ and $a'' = a - a' = 2a - b$. Since $|MC| = b$ and $|MB| = a$, we have $|BC| = b - a = a'$. The edges $BC$ and $CD$ make an edge of an $a \times a$ square when it folds to a small cube, hence $|CD| = a - a' = a''$. Since triangle $NBC$ is congruent to $CDE$, $|DE| = a'$, and hence $|EF| = a''$. Since the triangle $COJ$ is congruent to the right triangle $xyz$, we obtain $|CO| = a$, $|DO| = a'$, and hence $|EK| = a'$. Since $|EF| = a''$ and $|KJ| = a$, we have $|GH| = |MA| = a'$. Thus $|AC| = b - a' = a$. Therefore, the zig-zag line $ACDE$ can be glued to the zig-zag line $GFEK$ since all lengths are matched and they are orthogonal. By the fact $|LK| = b$, the resulting rectangle $LKJH$ should be a square by the area constraint for the belt.

The second case $2a < b$ is omitted, however, a similar idea works. In both cases, we have the theorem. □

By Theorem 4, we have the following immediately.

**Corollary 5** There are infinitely many sets of five polyominoes such that (1) the polyominoes can form a net of $c \times c \times c$ cube, and (2) they can form two nets of two cubes of size $a \times a \times a$ and $b \times b \times b$.

We remark that it is open that if there are infinitely many distinct non-regular rep-cubes.

**References**


