Rep-cubes: Unfolding and Dissection of Cubes

Dawei Xu* Takashi Horiyama† Ryuhei Uehara*

Abstract

Last year, a new notion of *rep-cube* was proposed. A rep-cube is a polyomino that is a net of a cube, and it can be divided into some polyominoes such that each of them can be folded to a cube. This notion was inspired by the notions of *polyomino* and *rep-tile*, which were introduced by Solomon W. Golomb. It was proved that there are infinitely many distinct rep-cubes. In this paper, we investigate this new notion and obtain three new results. First, we prove that there does not exist a regular rep-cube of order 3, which solves an open question proposed in the paper. Next, we enumerate all regular rep-cubes of order 2 and 4. For example, there are 33 rep-cubes of order 2; that is, there are 33 dodecominoes that can fold to a cube of size $\sqrt{2}\times\sqrt{2}\times\sqrt{2}$ and each of them can be divided into two nets of unit cube. Similarly, there are 7185 rep-cubes of order 4. Lastly, we focus on pythagorean triples that consist of three positive integers (a,b,c) with $a^2+b^2=c^2$. For each of these triples, we can consider a rep-cube problem that asks whether a net of a cube of size $c\times c\times c$ can be divided into two nets of two cubes of size $a\times a\times a$ and $b\times b\times b$. We give a partial answer to this natural open question by dividing into more than two pieces. For any given pythagorean triple (a,b,c), we construct *five* polyominoes that form a net of a cube of size $c\times c\times c$ and two nets of two cubes of size $a\times a\times a$ and $b\times b\times b$.

1 Introduction

A *polyomino* is a "simply connected" set of unit squares introduced by Solomon W. Golomb in 1954 [7]. Since then, polyominoes have been playing an important role in recreational mathematics (see, e.g., [5]). In 1962, Golomb also proposed an interesting notion called "*rep-tile*": a polygon is a rep-tile of order k if it can be divided into k replicas congruent to one another and similar to the original (see [6, Chap 19]).

From these notions, Abel et al. proposed a new notion [1]; a polyomino is said to be a *rep-cube* of order k if it is a net of a cube (or, it can fold to a cube), and it can be divided into k polyominoes such that each of them can fold to a cube. If all k polyominoes have the same size, we call the original polyomino a *regular* rep-cube of order k. We note that crease lines are not necessarily along the edges of the polyomino. For example, a regular rep-cube of order 2 folds to a cube by folding along the diagonals of unit squares; see Figure 1.

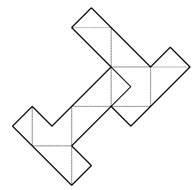

Figure 1: A regular rep-cube of order 2 [1]; each `T` shape can fold to a cube, and this shape itself can fold to a cube of size $\sqrt{2}\times\sqrt{2}\times\sqrt{2}$ by folding along the dotted lines.

In [1], Abel et al. propose regular rep-cubes of order k for each $k=2,4,5,8,9,36,50,64$, and also $k=36gk'^2$ for any positive integer k' and an integer g in $\{2,4,5,8,9,36,50,64\}$. In other words, there are infinitely many k that allow regular rep-cube of order k. On the other hand, they left an open problem that asks if there is a rep-cube of order 3. In this paper, we first answer to this question. There are no regular rep-cube of order 3.

Next we enumerate all possible regular rep-cubes of order k for small k. We mention that the following problem is not so easy to solve efficiently; for a given polygon P, determine if P can fold to a cube or not. Recently, Horiyama and Mizunashi developed an efficient algorithm that solves this problem for a given orthogonal polygon, which runs in $O((n+m)\log n)$ time, where n is the number of vertices in P, and m is the maximum number of line segments that appears on a crease line [8]. We remark that the parameter m is hidden and can be huge comparing to n. In our case, P is a polyomino, and this hidden parameter is linear to the number of unit squares in P, and hence our algorithm is simpler.

Finally, we investigate non-regular rep-cube. In [1], Abel et al. also asked if there exists a rep-cube of area 150 that is a net of a cube of size $5\times5\times5$ and it can be divided into two nets of cubes of size $3\times3\times3$ and $4\times4\times4$. This idea comes from a pythagorean triple $(3,4,5)$ with $3^2+4^2=5^2$. We give a partial answer to this question by dividing into more pieces than 2. We give a general way for any pythagorean triple (a,b,c) with $a<b<c$ to obtain five piece solution. That is, for any given pythagorean triple (a,b,c) with $a<b<c$, we construct a polyomino that is a net of a cube of $c\times c\times c$, and it can be divided into 5 pieces such that one of 5 pieces can fold to a cube of $a\times a\times a$, and gluing the

*School of Information Science, Japan Advanced Institute of Science and Technology, Japan. {xudawei,uehara}@jaist.ac.jp

†Graduate School of Science and Engineering, Saitama University, Japan. horiyama@al.ics.saitama-u.ac.jp

remaining 4 pieces, we can obtain a net of a cube of $b \times b \times b$.

Due to lack of space, some proofs are omitted.

2 Nonexistence of regular rep-cubes

The main theorem in this section is the following.

Theorem 1 *There does not exist a regular rep-cube of order* 3.

We first show two lemmas (proofs are omitted):

Lemma 2 *Let Q be a cube and P any development*[1] *of Q. Then P is concave.*

Let P be a polyomino (not necessarily hexomino) that can fold to a cube Q. Then, by Lemma 2, P has no "rolling belt" (see [4] for further details). This fact implies that, when we fold P to Q, each vertex on Q should appear at either the grid point of P or the middle point of a unit edge in P. For these vertices of Q, we state stronger property:

Lemma 3 *Let P be a polyomino that can fold to a cube Q. Let ℓ be the length of an edge of Q. (That is, P is a $6\ell^2$-omino.) Then P can be placed on a grid of size ℓ so that every vertex of Q on P is on a grid point. I.e., not only all vertices on Q appear on the boundary of P, but also they are also aligned on the grid points of size ℓ.*

Now we turn to the proof of Theorem 1. We assume that there exists a regular rep-cube of order 3, and derive contradictions. That is, we assume that there is a polyomino $\hat{P}$ such that $\hat{P}$ can be divided into three polyominoes P_1, P_2, P_3 of the same size, and each of $\hat{P}, P_1, P_2, P_3$ can fold to a cube of certain size. Let $\hat{Q}$ and Q_i denote the cubes folded from $\hat{P}$ and P_i, respectively. We suppose that the length of an edge of Q_i is ℓ. That is, P_i is a $6\ell^2$-omino, and $\hat{P}$ is a $18\ell^2$-omino. We remark that ℓ is not necessarily an integer, but $6\ell^2$ is.

Now we consider the polyomino P_1; that is a $6\ell^2$-omino, and folds to the cube Q_1 of size $\ell \times \ell \times \ell$. Then, by Lemma 3, P_1 can be on the grid of size ℓ so that every vertex of Q_1 is on the grid. We take any two vertices v_1 and v_2 of Q_1 of distance ℓ on the grid. Then the vector $\overrightarrow{v_1v_2}$ can be represented by (a, b) for some nonnegative integers a and b. That is, $a^2+b^2 = \ell^2$ for some integers a and b. (The same idea can be found in [4, Ch. 5.1.1] and [3].) We can apply the same argument to $\hat{P}$ and $\hat{Q}$, and hence there are some nonnegative integers $\hat{a}$ and $\hat{b}$ such that $\hat{a}^2+\hat{b}^2 = 3\ell^2$. Thus we obtain $\hat{a}^2+\hat{b}^2 = 3(a^2+b^2)$.

Therefore, it is sufficient to show that there are no such integers. To derive a contradiction, we assume that we have $\hat{a}^2 + \hat{b}^2 = 3(a^2 + b^2)$, and they are the minimum integers with respect to the value of $\hat{a}^2 + \hat{b}^2$.

Now, for an integer i, $(3i \pm 1)^2 = 9i^2 \pm 6i + 1$. Therefore, a square number x is either $x = 3x'$ or $3x' + 1$ for some integer x'. Since $\hat{a}^2 + \hat{b}^2 = 3(a^2 + b^2)$ is a multiple of 3, both of $\hat{a}$ and $\hat{b}$ are multiples of 3, say $\hat{a} = 3\hat{a}'$ and $\hat{b} = 3\hat{b}'$. Then, we have $(3\hat{a}')^2 + (3\hat{b}')^2 = 9(\hat{a}'^2 + \hat{b}'^2) = 3(a^2 + b^2)$. Thus we obtain $a^2 + b^2 = 3(\hat{a}'^2 + \hat{b}'^2)$. This contradicts the minimality of the value of $\hat{a}^2+\hat{b}^2$. Therefore, we have no such integers $a, b, \hat{a}, \hat{b}$. This completes the proof of Theorem 1. □

3 Enumeration of regular rep-cubes

In this section, we describe the exhaustive search algorithm for generating all regular rep-cubes of order k (for $k = 2$ and $k = 4$).

Algorithm 1 gives the outline of this algorithm. It works as follows: Let S_i be the set of all $(6 \times i)$-ominoes such that (1) it is composed by i nets of a unit cube, (2) it can cover a part of a cube of size $\sqrt{k} \times \sqrt{k} \times \sqrt{k}$. In the term of search of development, each element in S_i is called a *partial development* of a cube of size $\sqrt{k} \times \sqrt{k} \times \sqrt{k}$ [10]. That is, S_1 is the set of all nets of a unit cube, which consists of 11 hexominoes, and each set S_i with $i > 1$ is a subset of $(6 \times i)$-ominoes that can be computed from S_{i-1}. Let P_i be any polyomino in S_i, e.g., P_1 is one of the 11 hexominoes in S_1.

In Procedure CheckCover, the algorithm checks if P_i can cover the cube of size $\sqrt{k}\times\sqrt{k}\times\sqrt{k}$ without overlap. The details will be described later. Our final goal is to obtain the set S_k that contains all regular rep-cubes of order k from the set S_1.

Algorithm 1: Outline of the exhaustive search algorithm.

Input : Integer k of the order for the rep-cube;
Output: All rep-cubes in S_k;
for $i = 2$ *to* k **do**
  foreach *partial development* P_{i-1} *in* S_{i-1} **do**
    foreach *development* P_1 *in* S_1 **do**
      attach P_1 to P_{i-1} at each possible adjacency square on the boundary of P_{i-1} to obtain a new polyomino P_i;
      if *CheckCover(P_i)==1* **then**
        store P_i into S_i; // P_i is a partial
          // development of the box of
          // size $\sqrt{k} \times \sqrt{k} \times \sqrt{k}$
return S_k;

The algorithm works in a loop as follows. It picks up a polyomino P_{i-1} in S_{i-1} and a hexomino P_1 in

[1] We use "net" that has no overlap when it is spread out. When we use "development," overlap is not yet considered.

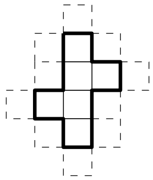

Figure 2: All possible adjacency empty squares on the boundary of a net.

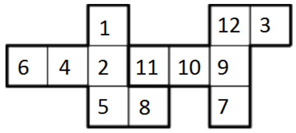

Figure 3: Every square of P is marked with a unique number according to the adjacency list.

S_1, and attaches P_1 by edge-to-edge gluing to P_{i-1} at each possible adjacency empty square on the boundary of P_{i-1} as shown in Figure 2. We note that we have to consider not only the overlap, but also the flip of P_1 if P_1 is not congruent to its mirror image. By this step, it generates a new polyomino P_i, which is a component of i nets of a unit cube. This P_i will be examined whether it can fold to a part of the cube of size $\sqrt{k} \times \sqrt{k} \times \sqrt{k}$ or not. This loop terminates at $i = k$, when the polyomino P_k can fold to a complete cube.

As mentioned in Introduction, we find that the folding lines of the cube of $\sqrt{k}\times\sqrt{k}\times\sqrt{k}$ are not along the edges of unit squares. Since the rep-cubes of order 2 and 4 have different folding ways, we need a universal method to check whether a polyomino is a partial development or not. In [10], the authors proposed an algorithm that checks the positional relationships of unit squares on the polyomino. Consider any polyhedron, e.g., a cube Q, folded from a polyomino P. Then we can obtain an adjacency relationship of unit squares in P on Q. That is, two unit squares share an edge on P only if they share it on Q. Thus any development of Q keeps a part of the same adjacency relationship. Therefore, we can decide if a polyomino P can fold to a cube Q by checking the positional relationship of the unit squares in Procedure CheckCover.

We consider the first development in Figure 5 as an example P (Figure 3). We first mark a unit square with the number 1 as the start point. Then we mark all of its neighbor-squares a number according to the adjacency list of cube of size $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$ as in Table 1 and Figure 4 in all four directions. (For example, the square 1 is surrounded by 12(above), 11(right), 2(below), and 3(left) from the viewpoint of the square 1.) This step is extended to its farther neighbors until every square of P is marked with a number. After this step, if every square in connected P is marked with its unique number, P can wrap the cube of size $\sqrt{k}\times\sqrt{k}\times\sqrt{k}$ with consistency. On the other hand, if (1) one square is marked with different numbers by its neighbors or (2) two or more squares are marked with the same number, then an overlap occurs in this folding way of P. We check all possible start points and directions for each P.

Table 1: Adjacency list of cube of size $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$.

Square ID	Up	Right	Down	Left
1	12	11	2	3
2	1	11	5	4
3	12	1	4	6
4	3	2	5	6
5	4	2	8	7
6	3	4	7	9
7	6	5	8	9
8	7	5	11	10
9	6	7	10	12
10	9	8	11	12
11	10	8	2	1
12	9	10	1	3

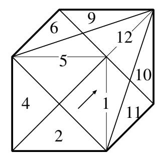

Figure 4: Adjacency relationship of the squares on the cube of size $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$.

Procedure CheckCover(P_i)

Input : Polyomino P_i in S_i;
Output: Whether P_i can wrap up the cube of size $\sqrt{k} \times \sqrt{k} \times \sqrt{k}$ or not;

```
foreach square in P_i do
    mark the square 1 as the start point
    foreach marked square in P_i do
        mark its unmarked adjacent squares as the
        adjacency matrix of the cube of size
        √k × √k × √k;
        if any square of P_i gets marked by two or
        more different numbers then
            break; // P_i has overlap
    if every square of P_i is marked by a unique
    number then
        return 1; // P_i can wrap up the cube
return 0;
```

As a result of finding the rep-cube of order 2, by putting two developments of a cube aside, there are 2424 distinct dodecominoes. Among them, there are 33 regular rep-cubes of order 2 that can fold to a cube of size $\sqrt{2} \times \sqrt{2} \times \sqrt{2}$ and each of them can be divided into two nets of a unit cube. As shown in Figures 5 and 6, we can observe that 17 rep-cubes out of 33 consist of two nets of the same shape. We call them *uniform* rep-cubes. Precisely, we say a regular rep-cube of order k is *uniform* if its all k nets are the same shape.

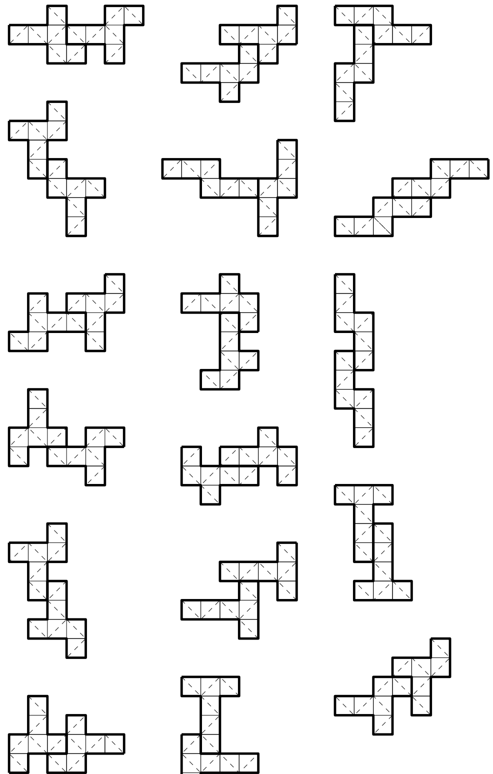

Figure 5: All 17 *uniform* rep-cubes of order 2.

For the case of finding the regular rep-cube of order 4, we also implement this algorithm. As a result, we got the amount of partial developments of i pieces as in Table 2, which means there are 7185 regular rep-cubes of order 4. Among them, we also find all uniform rep-cubes of order 4, which are 158 in total. One example of these uniform rep-cubes is shown in Figure 7. Out of 158, 98 of these uniform rep-cubes are made of pieces in shape (b) shown in Figure 8.

Table 2: The number of partial developments of regular rep-cubes of order 4.

Set of partial developments	S_1	S_2	S_3	S_4
Number of developments	11	2345	114852	7185

In Figure 1 of [1], they gave three uniform rep-cubes of order 2 (Figure 1), 4, and 9. On the other hand, in [1], they also show a regular rep-cube of order 50 that contains all kinds of 11 nets of a unit cube. It may worth focusing on these special cases for a larger k.

In the analysis of the results, we found two different patterns of shapes that can make the same rep-cube. As shown in Figure 9, except for the difference in composition, these two rep-cubes have the same contour, the same surface area and the same folding way. Finding this kind of rep-cube can be a interesting topic in the future research.

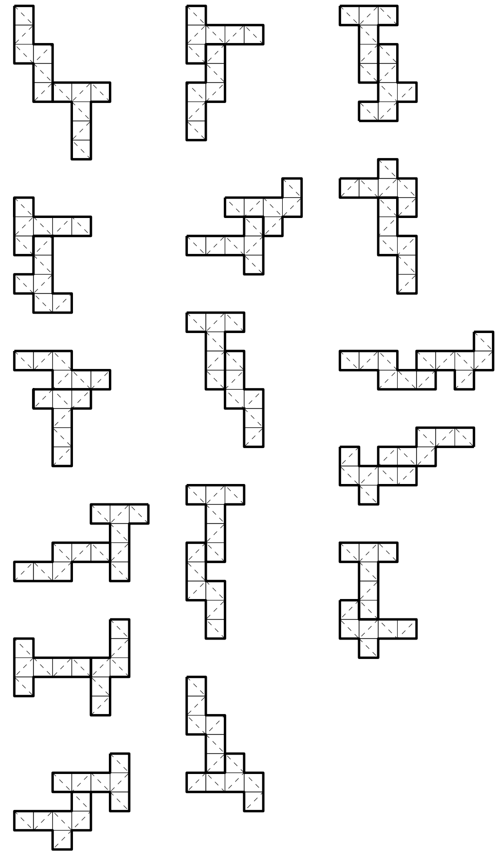

Figure 6: All regular rep-cubes of order 2 that are not *uniform*.

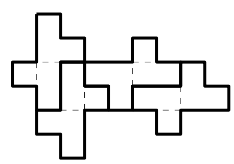

Figure 7: An example of *uniform* rep-cubes of order 4.

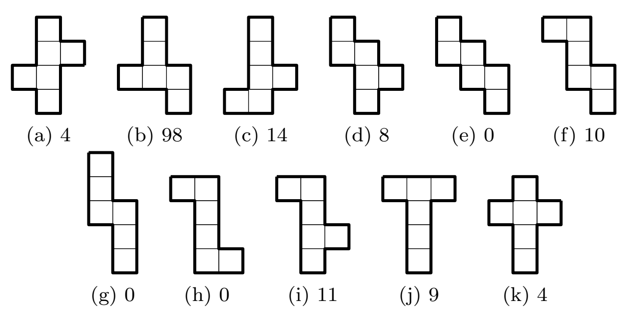

Figure 8: List of the amount of uniform rep-cubes of order 4 made by each of 11 shapes.

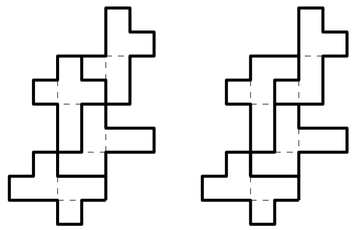

Figure 9: Two different patterns make the same rep-cube of order 4.

4 Rep-cubes based on pythagorean triples

A *pythagorean triple* is a 3-tuple of positive integers that satisfies $a^2 + b^2 = c^2$. In [1], Abel et al. propose an interesting open question related to the pythagorean triple. That is, the question asks whether there is a rep-cube of order 2 of area c^2 such that (1) it folds to a $c \times c \times c$ cube, and (2) it can be divided into two polyominoes so that they fold to a $a \times a \times a$ cube and another $b \times b \times b$ cube. The most famous one is $(3, 4, 5)$ with $3^2 + 4^2 = 5^2$. We note that for any pythagorean triple (a, b, c), for any positive integer k, (ak, bk, ck) is also a pythagorean triple. However, we only consider pythagorean triples with $GCD(a, b, c) = 1$. Then, it is known that a triple (a, b, c) with $GCD(a, b, c) = 1$ is a pythagorean triple if and only if there are two positive integers m, n such that m, n are relatively prime, $0 < n < m$, $m - n$ is odd, and we can obtain a pythagorean triple as $(m^2 - n^2, 2mn, m^2 + n^2)$ for these n and m.

It is trivial that when we divide any net of a $c \times c \times c$ cube into $6c^2$ unit squares, we can make two cubes of size $a \times a \times a$ and $b \times b \times b$. Therefore, we can consider this open problem as an optimization problem to minimize the number of polyominoes that can form both of a net of $c \times c \times c$ cube, and two nets of two cubes of size $a \times a \times a$ and $b \times b \times b$. In this section, we give the following theorem:

Theorem 4 *Let (a, b, c) be any pythagorean triple with $a < b < c$. Then we can construct a set $S(a, b, c)$ of five polyominoes such that (1) the polyominoes can form a net of $c \times c \times c$ cube, and (2) they can form two nets of two cubes of size $a \times a \times a$ and $b \times b \times b$.*

We here show an example in Figure 10 to get the idea. When we choose a pythagorean triple $(3, 4, 5)$, the polyomino in Figure 10(a) folds to a $3 \times 3 \times 3$ cube, and the polyomino in Figure 10(b) folds to a $4 \times 4 \times 4$ cube. It is less intuitive, however, the reader can obtain a $5 \times 5 \times 5$ cube from the polyomino in Figure 10(c) by folding along the dotted lines. Here we give a general construction for any pythagorean triple.

Proof. We first give a brief idea of the construction in Figure 11. The first step is that we open two small cubes

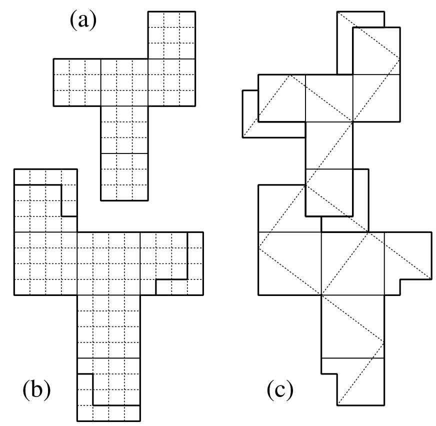

Figure 10: The set $S(3, 4, 5)$ of five polyominoes that folds to (a,b) two cubes of size $3 \times 3 \times 3$ and $4 \times 4 \times 4$, and (c) one cube of size $5 \times 5 \times 5$.

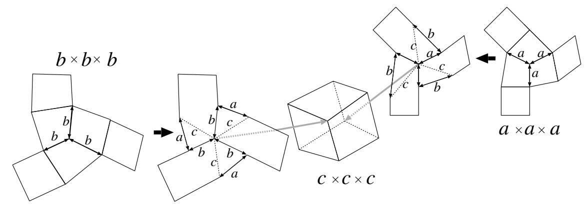

Figure 11: Brief idea of the construction.

of size $a \times a \times a$ and $b \times b \times b$ at their any vertices. We cut along the three lines from the vertex until we have a kind of a triangular-pyramid-like shape; each rectangular face consists of two squares, and these three rectangles are glued like in wind-wheel shape. Then we regard these two triangular pyramids as cone-like shapes, and attach each of apexes to the two opposite vertices of the big cube of size $c \times c \times c$. That is, they are glued to two endpoints of a diagonal of the big cube.

The main trick is that the grids of two small cubes are not aligned to the grid of the big cube; we twist two cones so that their edges (or grid lines) make two edges of pythagorean triangle of length a and b. As a result, three vertices of the big cube are on the boundary of the wind-wheel shape made from the cube of size $b \times b \times b$, and the other three vertices of the big cube are on the boundary of the other wind-wheel shape made from the cube of size $a \times a \times a$. Then, we have two cases depending on the size of these two small cubes.

The first case is that $a < b < 2a$. For example, the most famous pythagorean triple $(3, 4, 5)$ (for $m = 2, n = 1$) satisfies this condition. In this case, the situation is illustrated on the net of the big cube in Figure 12. The outline is the net of the big cube, and three vertices labeled by p form a vertex of the big cube, and the apex of the cone made by the small cube of size $a \times a \times a$

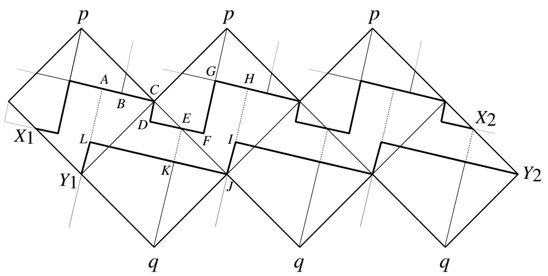

Figure 12: View on the net of the big cube of size $c\times c\times c$.

is attached at the vertex p. In the figure, all squares of this small cube are already depicted, and they are aligned along the zig-zag line joining two points X_1 and X_2. On the other side, three vertices labeled by q form the opposite vertex of the big cube, where the apex of the cone made by the other small $b \times b \times b$ cube is attached to. In the figure, three squares of size $b \times b$ are depicted along the zig-zag line joining two points Y_1 and Y_2. Therefore, out task is to form three more squares of size $b \times b$ by the belt between lines X_1X_2 and Y_1Y_2 with few dissections.

We first extend the grid lines of squares of size $b \times b$ as shown in Figure 12. Then the belt is divided into six parts; three of them are congruent to the hexagon $ACDEKL$, and three of them are congruent to the hexagon $EFGHJK$. Then our claim is that gluing the line $GFEK$ to $ACDE$, we obtain a square $HJKL$ of size $b\times b$. If it works, it is easy to see the theorem holds.

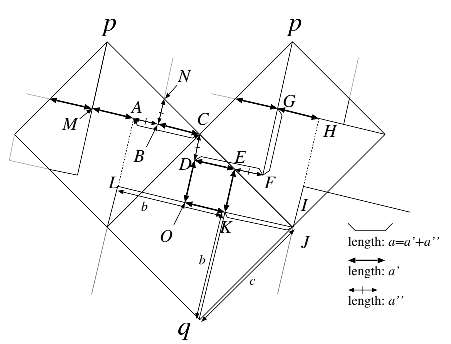

Figure 13: Detailed lengths of polyominoes.

Now we focus on this part (see Figure 13). We first observe that two triangles pMC and JKq are congruent to the right triangle xyz with $|xy| = a$, $|yz| = b$, and $|zx| = c$. We now let $a' = b-a$ and $a'' = a-a' = 2a-b$. Since $|MC| = b$ and $|MB| = a$, we have $|BC| = b-a = a'$. The edges BC and CD make an edge of an $a \times a$ square when it folds to a small cube, hence $|CD| = a-a' = a''$. Since triangle NBC is congruent to CDE, $|DE| = a'$, and hence $|EF| = a''$. Since the triangle COJ is congruent to the right triangle xyz, we obtain $|CO| = a$, $|DO| = a'$, and hence $|EK| = a'$. Since $|EF| = a''$ and $|KJ| = a$, we have $|GH| = |MA| = a'$. Thus $|AC| = b - a' = a$. Therefore, the zig-zag line $ACDE$ can be glued to the zig-zag line $GFEK$ since all lengths are matched and they are orthogonal. By the fact $|LK| = b$, the resulting rectangle $LKJH$ should be square by the area constraint for the belt.

The second case $2a < b$ is omitted, however, a similar idea works. In both cases, we have the theorem. □

By Theorem 4, we have the following immediately.

Corollary 5 *There are infinitely many sets of five polyominoes such that (1) the polyominoes can form a net of $c \times c \times c$ cube, and (2) they can form two nets of two cubes of size $a \times a \times a$ and $b \times b \times b$.*

We remark that it is open that if there are infinitely many distinct non-regular rep-cubes.

References

[1] Z. Abel, B. Ballinger, E. D. Demaine, M. L. Demaine, J. Erickson, A. Hesterberg, H. Ito, I. Kostitsyna, J. Lynch, and R. Uehara. Unfolding and Dissection of Multiple Cubes. In *JCDCG*^{3}, 2016.

[2] J. Akiyama. Tile-Makers and Semi-Tile-Makers. *American Mathematical Monthly*, Vol. 114, pp. 602–609, 2007.

[3] Y. Araki, T. Horiyama, and R. Uehara. Common Unfolding of Regular Tetrahedron and Johnson-Zalgaller Solid. *J. of Graph Algorithms and Applications*, Vol. 20, No. 1, pp. 101–114, 2016.

[4] E. D. Demaine and J. O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra*. Cambridge, 2007.

[5] M. Gardner. *Hexaflexagons, Probalitity Paradoxes, and the Tower of Hanoi*. Cambridge, 2008.

[6] M. Gardner. *Knots and Borromean Rings, Rep-Tiles, and Eight Queens*. Cambridge, 2014.

[7] S. W. Golomb. *Polyominoes: Puzzles, Patterns, Problems, and Packings*. Princeton Univ., 1996.

[8] T. Horiyama and K. Mizunashi. Folding Orthogonal Polygons into Rectangular Boxes. *19th Korea-Japan Joint Workshop on Algorithms and Computation*, 2016.

[9] J. Mitani and R. Uehara. Polygons Folding to Plural Incongruent Orthogonal Boxes, *CCCG 2008*, pp. 39-42, 2008.

[10] D. Xu, T. Horiyama, T. Shirakawa, and R. Uehara. Common Developments of Three Incongruent Boxes of Area 30. *COMPUTATIONAL GEOMETRY: Theory and Applications*, Vol. 64, pp. 1–17, 2017.